TECHNISCHE UNIVERSITÄT CAROLO-WILHELMINA ZU BRAUNSCHWEIG

Studienarbeit

Asynchronous Petri Net Classes

Jens-Wolfhard Schicke

24. April 2008

Institut für Programmierung und Reaktive Systeme
Prof. Dr. Ursula Goltz

betreut durch:

Prof. Dr. Ursula Goltz
Prof. Dr. Rob van Glabbeek

Abstract

Keywords Asynchrony, Synchrony, Petri Net Classes, Semantical Characterization

Contents

List of Figures

List of Abbreviations

AA(H,B)	tail asymmetrically asynchronous nets respecting branching time
AA(H,L)	tail asymmetrically asynchronous nets respecting linear time
AA(M,B)	asymmetrically asynchronous nets respecting branching time
AA(M,L)	asymmetrically asynchronous nets respecting linear time
AA(V,B)	front asymmetrically asynchronous nets respecting branching time
AA(V,L)	front asymmetrically asynchronous nets respecting linear time
AI	asymmetrically asynchronous implementation
BFC	behaviourally free choice
EFC	extended free choice nets
ESPL	extended simple nets
FC	free choice nets
FSA(B)	fully symmetrically asynchronous nets respecting branching time
FSA(L)	fully symmetrically asynchronous nets respecting linear time
FSI	fully symmetrically implementation
SA(B)	symmetrically asynchronous nets respecting branching time
SA(L)	symmetrically asynchronous nets respecting linear time
SI	symmetrically implementation
SPL	simple nets
TSPL	simple nets in terms of transitions

1 Introduction

The objective of this paper is to describe the distinction between synchrony and asynchrony in distributed systems in novel and detailed ways.

Naively, synchrony between two events in a distributed system means that both events happen "at the same time". In real-world systems however this concept is ill-defined as the speed of light introduces some inherent amount of asynchrony everywhere in the system and whether two events happen at the same time depends on the observer. Nonetheless two events can be considered synchronous when "nothing of importance could have happened between them."

Consider for example two concurrently running processes A and B which wish to exchange information by sending some kind of message. The event of A sending the message and the event of B receiving it can now either be synchronous or asynchronous. If the two events happen synchronously, no further computation can happen anywhere in the system while the message travels, which in particular means that B is indeed ready to receive the message when A sends it. If however the two events are asynchronous, B might decide, after A sent the message, not to communicate and instead do something else. Thus it is not guaranteed that B will ever receive the message as intended by A.

In practice, to get a system in which synchrony between events is meaningful clocks are used, as seen in many computer chips. However the larger the part of the system, which is synchronized using the same clock, the lower the performance will be. Thus splitting the system in many asynchronous parts will improve performance, sometimes considerably.

To help in this splitting, we want to answer the question which events in a system are asynchronous, that is whether they occur synchronously or not with other events is irrelevant.

Much has already been written about related questions during the last decades. Using variations of CSP there are [5], [6] and [7], Petri nets have been covered in [10], locally synchronous systems in general by [2] and recently asynchronous π-calculus has been employed by [9], [16] and [15].

Impossibility results for encoding synchrony in asynchronous systems have been obtained in some of these papers while other ones achieved concrete encodings for the same problem using other constraints.

More hardware oriented results exist as well, as the problem of how to implement a specified behaviour using the most performant communication possible frequently occurs during chip design. See [12] for some examples.

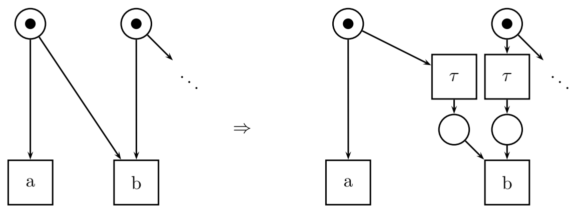

Figure 1.1: Transformation to the symmetrically asynchronous implementation

An overview and a detailed comparison between our results and the literature is made in Section 6.

To study the problem in a basic model independent from specific language constructs, we have chosen Petri nets as our model of computation.

In Petri nets, only very low-level primitives are available and the differences between synchrony and asynchrony are hence more obvious. As do many other formal models, Petri nets have, despite their rather small set of primitives, synchrony already built in: Whenever a transition fires, the tokens of all preplaces are removed atomically, and no other transition can use them. This becomes especially significant in the case of conflict, where multiple transitions share the same preplace. To disallow this form of synchrony and get an “asynchronous” Petri net, the reality of physical processes can be mimicked in the form of silent transitions which pretend that removing tokens is not an instantaneous action. Thus other events can occur even while one transition is in the process of firing.

We call the net with the newly introduced transitions an “implementation”, as it represents a possible real-world implementation of the original net. In this paper we introduce three different possible asynchronous implementations, namely the “fully symmetrically asynchronous”, the “symmetrically asynchronous” and the “asymmetrically asynchronous” implementation, which differ in how much additional structure is allowed between the invisible transitions to manage the removal of tokens. These different implementations represent different grades of asynchrony, thereby enabling us to describe which communication structures can still be implemented at which grade of asynchrony.

An example of such an asynchronous implementation of a net can be seen in Figure 1.1. It can be seen as a representation of two processes (one on each side of the net) which communicate synchronously by executing the transition b together. Note that after in the implementation the sender can no longer be sure whether the receiver will ever be willing to process the message, which was not the case before.

The new system can still perform the same set of actions, but can also deadlock. These two behaviours seem intuitively different. To formalize this intuition of difference, equivalence relations are used, which define when exactly two systems are “the same”.

A quite comprehensive overview over existing equivalence relations for reactive systems is given by [19], [20] and [21]. Such equivalence relations can be classified along different dimensions, two of the most prominent being the sensitivity to the decision structure between alternative behaviours of a system and the sensitivity to causality between different actions. Along the first dimension, equivalences which essentially disregard the decision structure are called linear time equivalences whereas ones which respect it (in more or less detail) are called branching time equivalences.

Returning to our original problem, we can now characterise classes of Petri nets by considering whether they are equivalent to their implementation. This characterisation has two parameters we can choose: By which equivalence relation to compare the behaviour and how exactly to perform the implementation, i.e. where to insert new transitions and which structures to allow between the them. Choosing different sets of parameters will not only give new insight into the difference between synchrony and asynchrony but will also produce a classification of equivalence relations with respect to their ability to discern the two.

We will start our search for useful equivalence relations at the coarsest end of the spectrum, namely trace equivalence, comparing just the sequences of actions performed. It will turn out however, that neither trace equivalence nor completed trace equivalence is suited to our needs.

We finally find a useful "linear time" equivalence by comparing the pomsets of maximal processes of a net. This equivalence respects causality and parallelism and enables us to detect local deadlocks in spite of infinite concurrent activity. Since parallelism is respected we can argue that the implementation will be "as efficient" as the original net.

For branching time semantics, we use failures equivalence which is one of the most used equivalences.

It turns out that our semantically characterised net classes, induced by the various implementations and equivalence relations, are related to well known structural net classes. Symmetrically asynchronous nets relate to free-choice and extended free choice nets, while asymmetrically asynchronous nets relate to simple nets. The exact relations naturally depend on the chosen equivalence relation. This result implies that free choice and simple nets can be easily distributed. Our classes are larger than the structural ones however, as distributability depends on concrete behaviour and not static structure.

In Section 2, we proceed by introducing some basic notions necessary for the subsequent examination of net classes. Afterwards Section 3 describes the effects of the fully symmetrically asynchronous implementation, first by proving some basic lemmas about the implementations behaviour then by giving a more structural characterisation of one of the resulting net classes. Section 4 then repeats those steps for the symmetrically asynchronous implementation. Additionally relations to various structural net classes are given. In Section 5 those two steps are also done for the asymmetrically asynchronous implementation. Finally an analysis of how the results of related work are connected with ours is given in the conclusions in Section 6.

2 Basic Notions

We consider here 1-safe net systems, i.e. places never carry more than one token and a transition can fire even if pre- and postset intersect. To represent unobservable behaviour, which we use to model asynchrony, the set of transitions is partitioned into observable and unobservable ones.

Definition 2.1
A *net with silent transitions* is defined as $N = (S, O, U, F, M_0)$ where

- S is a set (of *places*),
- O is a set (of *observable transitions*),
- U is a set (of *unobservable transitions*),
- $F \subseteq (S \times T \cup T \times S)$ (the *flow relation*) with $T = O \cup U$ (*transitions*) and
- $M_0 \subseteq S$ (the *initial marking*).

In this paper we only consider finite nets, i.e. S, O, U are all finite.

We denote the preset and postset of a net element x by ${}^\bullet x := \{y \mid (y, x) \in F\}$ and by $x^\bullet := \{y \mid (x, y) \in F\}$ respectively. Where necessary we extend functions to sets element-wise. Furthermore the transitive closure of the flow relation is denoted F^+.

The semantics of such a Petri net can be described using the "token game": Whenever all preplaces of a transition hold a token (i.e. ${}^\bullet x \subseteq M$) that transition can fire, thereby removing all those tokens and generating new ones on its post-places.

Definition 2.2 Let $N = (S, O, U, F, M_0)$ be a net. Let $M_1, M_2 \subseteq S$.
$G \subseteq (O \cup U), G \neq \varnothing$, is called a *step from* M_1 *to* M_2, $M_1[G\rangle_N M_2$, iff

- all transitions contained in G are enabled, i.e.

$$\forall t \in G.\ {}^\bullet t \subseteq M_1 \wedge (M_1 \setminus {}^\bullet t) \cap t^\bullet = \varnothing\ ,$$

- all transitions of G are independent, that is not conflicting:

$$\forall t, u \in G, t \neq u.\ {}^\bullet t \cap {}^\bullet u = \varnothing \wedge t^\bullet \cap u^\bullet = \varnothing\ ,$$

- in M_2 all tokens have been removed from the preconditions and new tokens have been inserted at the postconditions:

$$M_2 = \left(M_1 \setminus \bigcup_{t \in G} {}^\bullet t\right) \cup \bigcup_{t \in G} t^\bullet\ .$$

We omit the subscript N if clear from context. To make proofs about contact freeness easier in notation, we introduce a notation for a possibly not contact-free step and write $M_1[G)M_2$ iff $\forall t \in G.\ {}^\bullet t \subseteq M_1$ and the second and third conditions from above hold.

To simplify statements about possible behaviours of a net, we introduce some abbreviations.

Definition 2.3
For a net $N = (S, O, U, F, M_0)$, we define three relations:

- $\longrightarrow_N \subseteq \mathcal{P}(S) \times \mathcal{P}(O) \times \mathcal{P}(S)$ as $M_1 \xrightarrow{G}_N M_2 \Leftrightarrow G \subseteq O \wedge M_1[G\rangle_N M_2$
- $\xrightarrow{\tau}_N \subseteq \mathcal{P}(S) \times \mathcal{P}(S)$ as $M_1 \xrightarrow{\tau}_N M_2 \Leftrightarrow \exists t \in U.\ M_1[\{t\}\rangle_N M_2$
- $\Longrightarrow_N \subseteq \mathcal{P}(S) \times O^* \times \mathcal{P}(S)$ as $M_1 \stackrel{\sigma}{\Longrightarrow}_N M_2 \Leftrightarrow \exists n \geq 0.\ \sigma = t_1 t_2 \cdots t_n \subseteq O^* \wedge M_1 \xrightarrow{\tau}{}^*_N \xrightarrow{\{t_1\}}_N \xrightarrow{\tau}{}^*_N \xrightarrow{\{t_2\}}_N \xrightarrow{\tau}{}^*_N \cdots \xrightarrow{\tau}{}^*_N \xrightarrow{\{t_n\}}_N \xrightarrow{\tau}{}^*_N M_2$

We write $M_1 \xrightarrow{G}$ for $\exists M_2.\ M_1 \xrightarrow{G} M_2$, $M_1 \stackrel{G}{\not\longrightarrow}$ for $\nexists M_2.\ M_1 \xrightarrow{G} M_2$ and similar for the other two relations. We write $M_1 \not\longrightarrow$ for $M_1 \stackrel{\tau}{\not\longrightarrow} \wedge \forall G \subseteq O.\ M_1 \stackrel{G}{\not\longrightarrow}$.

A marking M_1 is said to be *reachable* iff there exists a $\sigma \in O^*$ such that $M_0 \stackrel{\sigma}{\Longrightarrow} M_1$. The set of all reachable markings is denoted by $[M_0\rangle$.

This paper only considers *contact-free* nets where in every reachable marking $M_1 \in [M_0\rangle$ for all $t \in O \cup U$ with ${}^\bullet t \subseteq M_1$

$$(M_1 \setminus {}^\bullet t) \cap t^\bullet = \varnothing\ .$$

Definition 2.4
A tuple $N = (S, O, U, F, M_0)$ is an *occurrence net* iff

- all conditions of Definition 2.1 hold,
- $\forall x, y \in S \cup O \cup U.\ (x, y) \in F^+ \Rightarrow (y, x) \notin F^+$,
- $\forall s \in S.\ |{}^\bullet s| \leq 1 \wedge |s^\bullet| \leq 1$ and
- $M_0 = \{s \mid s \in S, {}^\bullet s = \varnothing\}$.

A place $s \in S$ in an occurrence net is said to be *maximal* iff $s^\bullet = \varnothing$. We write N° for the set of all maximal places of an occurrence net N. Similarly we write ${}^\circ N$ for the set of minimal places defined as ${}^\circ N := \{s \mid s \in S, {}^\bullet s = \varnothing\}$. Note that we do not enforce finiteness for occurrence nets.

Definition 2.5
A *slice* of a net $N = (S, O, U, F, M_0)$ is a maximal set $C \subseteq S$ such that $\forall x, y \in C.\ (x, y) \notin F^+$.

Definition 2.6 Let $N = (S, O, U, F, M_0)$ be a net and let $N' = (S', O', U', F', M'_0)$ be an occurrence net.
A mapping $f : (S' \cup O' \cup U') \rightarrow (S \cup O \cup U)$ is a *process* of N iff

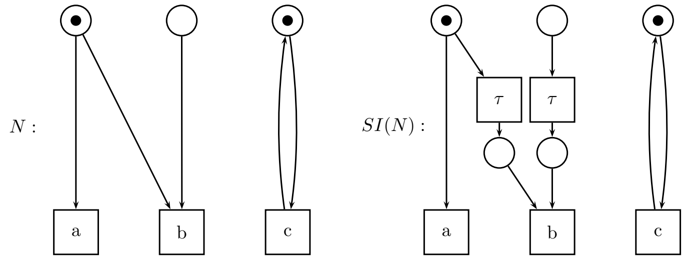

Figure 2.1: A net without completed traces

– $f(S') \subseteq S \wedge f(O') \subseteq O \wedge f(U') \subseteq U$,
– $f(M'_0) = M_0$,
– for every slice C of N', $\forall x, y \in C.\ f(x) = f(y) \Rightarrow x = y$ (f is injective over all slices) and
– $\forall t' \in O' \cup U'.\ f({}^\bullet t') = {}^\bullet f(t') \wedge f(t'^\bullet) = f(t')^\bullet$.

Definition 2.7

A process f from an occurrence net N' to a net N is said to be *maximal* iff $f(N'^\circ) \not\longrightarrow_N$. The set of all maximal processes of a net N is denoted by $MP(N)$.

To describe which nets are "asynchronous", we wish to compare their behaviour to that of their implementations using equivalence relations. The simplest useful equivalence available is trace equivalence. This equivalence declares two nets $N = (S, O, U, F, M_0)$ and $N' = (S', O', U', F', M'_0)$ to be equivalent iff every trace of either net is always also possible in the other, i.e. $\forall \sigma.\ (M_0 \overset{\sigma}{\Longrightarrow}_N) \Leftrightarrow (M'_0 \overset{\sigma}{\Longrightarrow}_{N'})$. However we will find (in Lemma 4.4) that trace equivalence will always treat original and implementation as equivalent and we would thus be unable to discern synchronous and asynchronous nets.

The difference in behaviour between a net and its implementation will always be in the existence of deadlocks as in the example of Figure 1.1. To detect deadlocks, completed trace equivalence is usually used. This equivalence additionally compares whether a trace was complete, i.e. whether no further transition could fire after the net produced the trace – or in formal term, whether additionally $\forall \sigma.\ (M_0 \overset{\sigma}{\Longrightarrow}_N M_1 \wedge M_1 \overset{\tau}{\not\longrightarrow}_N) \Leftrightarrow (M'_0 \overset{\sigma}{\Longrightarrow}_{N'} M_1 \wedge M_1 \overset{\tau}{\not\longrightarrow}_{N'})$. However the example in Figure 2.1 should intuitively not be asynchronous as one component could deadlock in the implementation which nonetheless has the same completed traces as the original net, i.e. none. So completed trace equivalence won't provide the distinction we want either. We need some notion of justice, which forces transitions to fire ultimately if continuously enabled. As noted in [18], justice in linear time is best described using causality respecting equivalences.

Thus we will consider two nets equivalent if the sets of visible pomsets obtained from their respective maximal processes are equal. The resulting equivalence relation respects causality and parallelism and yields a just semantics.

Definition 2.8
A *labelled partial order* is a structure $(V, T, \leq, l)$ where

- V is a set (of *vertices*),
- T is a set (of *labels*),
- $\leq \subseteq V \times V$ is a partial order relation and
- $l : V \to T$ (the *labelling function*).

Two labelled partial orders $o = (V, T, \leq, l)$ and $o' = (V', T, \leq', l')$ are *isomorphic*, $o \approx o'$, iff there exist a bijection $\varphi : V \to V'$ such that

- $\forall v \in V.\ l(v) = l'(\varphi(v))$ and
- $\forall u, v \in V.\ u \leq v \Leftrightarrow \varphi(u) \leq' \varphi(v)$.

Definition 2.9 Let $o = (V, T, \leq, l)$ be a partial order.
The *pomset* of o is its isomorphism class $[o] := \{o' \mid o \approx o'\}$.

By hiding the unobservable transitions of a process, we gain a pomset which describes causality relations of all participating visible transitions.

Definition 2.10 Let $f : (S', O', U', F', M_0') \to (S, O, U, F, M_0)$ be a process.
The *visible pomset* of f is the pomset $VP(f) := [(O', O, F'^*, f \cap (O' \times O))]$ where F'^* is the transitive and reflexive closure of the flow relation F'.

$MVP(N) := \{VP(f) \mid f \in MP(N)\}$ is the set of pomsets of all maximal processes of N.

Definition 2.11
Two nets N and N' are *completed pomset trace equivalent*, $N \simeq_{CPT} N'$, if and only if $MVP(N) = MVP(N')$.

To consider branching time semantics, we use failures equivalence, which, while quite a coarse branching-time equivalence, is sufficient for our means. Since our construction does not introduce new causalities nor removes parallelism, finer branching time equivalences should not lead to different results later on.

Definition 2.12 Let $N = (S, O, U, F, M_0)$ be a net, $\sigma \in O^*$ and $X \subseteq O$.
$<\sigma, X>$ is a *failure pair* of N iff

$$\exists M_1.\ M_0 \overset{\sigma}{\Longrightarrow} M_1 \wedge M_1 \overset{\tau}{\not\longrightarrow} \wedge \forall t \in X.\ M_1 \overset{\{t\}}{\not\longrightarrow}\ .$$

We define $F(N) := \{<\sigma, X> \mid <\sigma, X> \text{ is a failure pair of } N\}$.

Definition 2.13
Two nets N and N' are *failures equivalent*, $N \simeq_F N'$, iff $F(N) = F(N')$.

The following lemma might seem obvious, but it is nonetheless important, as many of the later proofs depend on it.

Lemma 2.1 Let $N = (S, O, \varnothing, F, M_0)$ be a net (without silent transitions) and $M \subseteq S$.

$$\text{If } M \stackrel{\sigma}{\Longrightarrow} M_1 \wedge M \stackrel{\sigma}{\Longrightarrow} M_2 \text{ then } M_1 = M_2.$$

Proof Let $t \in O$.

$M \stackrel{\{t\}}{\Longrightarrow} M_1' \Leftrightarrow M \stackrel{\{t\}}{\longrightarrow} M_1'$ and $M \stackrel{\{t\}}{\longrightarrow} M_1' \Rightarrow M_1' = (M \setminus {}^\bullet t) \cup t^\bullet$.

Hence $M \stackrel{\{t\}}{\Longrightarrow} M_1' \wedge M \stackrel{\{t\}}{\Longrightarrow} M_2' \Rightarrow M_1' = M_2'$. The result follows for a trace σ by induction on the length of σ. $\square$

A net $N = (S, O, U, F, M_0)$ with silent transitions is called *divergence free* iff $\forall M_1 \in [M_0\rangle$ $\exists n \in \mathbb{N}\, \forall M_2, \ldots, M_n \subseteq S.\ (M_1 \stackrel{\tau}{\longrightarrow} M_2 \stackrel{\tau}{\longrightarrow} \cdots \stackrel{\tau}{\longrightarrow} M_n \Rightarrow M_n \stackrel{\tau}{\not\longrightarrow})$.

3 Fully Symmetric Asynchrony

To examine the difference between synchronous and asynchronous communication, we will give different possible definitions of how asynchronous communication can be modelled in Petri nets. A simple and intuitive method to do this is to insert invisible transitions between visible ones and their preplaces. This simulates that it may take time to remove a token.

Definition 3.1 Let $N = (S, O, \varnothing, F, M_0)$ be a net.
The *fully symmetrically asynchronous implementation* of N is defined as the net $\mathit{FSI}(N) := (S \cup S^\tau, O, U', F', M_0)$ with

$$\begin{aligned} S^\tau &:= \{s_t \mid t \in O, s \in {}^\bullet t\} \ , \\ U' &:= \{t_s \mid t \in O, s \in {}^\bullet t\} \text{ and} \\ F' &:= F \cap (O \times S) \\ &\quad \cup \{(s, t_s) \ \mid t \in O, s \in {}^\bullet t\} \\ &\quad \cup \{(t_s, s_t) \mid t \in O, s \in {}^\bullet t\} \\ &\quad \cup \{(s_t, t) \ \mid t \in O, s \in {}^\bullet t\} \ . \end{aligned}$$

We will use the abbreviations ${}^\circ x := \{y \mid (y, x) \in F'\}$ and $x^\circ := \{y \mid (x, y) \in F'\}$ instead of ${}^\bullet x$ or $x^\bullet$ when making assertions about the flow relation of an implementation.

To understand the behaviour of the implementation, we first describe the structure of the reachable markings therein. Whenever the implementation enables some transition, first some silent transitions must fire, thereby moving tokens from the original places onto the newly introduced buffering places. To undo those silent transitions and get back the

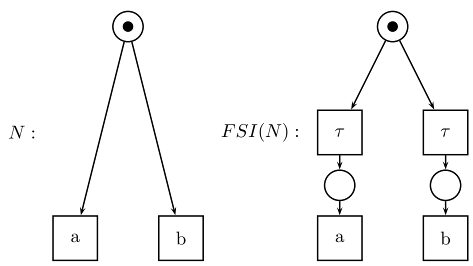

Figure 3.1: A net which is not failures equivalent to its fully symmetrically asynchronous implementation, $N \notin \mathit{FSA}(B)$, $N \in \mathit{SA}(B)$

original marking we define a function which maps markings of the implementation onto markings of the original net.

Definition 3.2 Let $N = (S, O, \varnothing, F, M_0)$ be a net, let $FSI(N) = (S \cup S^\tau, O, U', F', M_0)$.
Let $\tau^{\leftarrow} : S \cup S^\tau \to S$ be the function defined by

$$\tau^{\leftarrow}(p) := \begin{cases} s & \text{iff } p = s_t \text{ with } s_t \in S^\tau, s \in S, t \in O \\ p & \text{otherwise } (p = s \in S) \end{cases} .$$

Furthermore, we define a predicate which is true on all markings of an implementation which can be reached. Additionally we provide a distance function specifying how many silent transitions can be fired in sequence.

Definition 3.3 Let $N = (S, O, \varnothing, F, M_0)$ be a net and $FSI(N) = (S \cup S^\tau, O, U', F', M_0)$.
The predicate $\alpha \subseteq \mathcal{P}(S \cup S^\tau)$ is defined as $\alpha(M) :\Leftrightarrow \tau^{\leftarrow}(M) \in [M_0\rangle_N \wedge \forall p, q \in M.$ $\tau^{\leftarrow}(p) \neq \tau^{\leftarrow}(q)$. The function $d : \mathcal{P}(S \cup S^\tau) \to \mathbb{N}$ is defined as $d(M) := |M \cap \{s | s \in S,$ $s^\bullet \neq \varnothing\}|$.

Using these two definitions, we can now proceed to prove basic properties of how the implementation works.

Lemma 3.1 Let $N = (S, O, \varnothing, F, M_0)$ be a net, $FSI(N) = S \cup S^\tau, O, U', F', M_0)$ and $M \subseteq S \cup S^\tau$.

(i) $\alpha(M_0)$

(ii) $\alpha(M) \Rightarrow (d(M) > 0 \Leftrightarrow \exists M' \subseteq S \cup S^\tau.\ M \xrightarrow{\tau}_{FSI(N)} M')$

(iii) $M[G\rangle_{FSI(N)} M' \wedge \alpha(M) \Rightarrow \forall t \in G.\ (M \setminus {}^\circ t) \cap t^\circ = \varnothing \wedge \tau^{\leftarrow}(M) \xrightarrow{G \cap O}_N \tau^{\leftarrow}(M') \wedge$ $\alpha(M')$

(iv) $M \xrightarrow{\tau}_{FSI(N)} M' \Rightarrow d(M) > d(M') \wedge \tau^{\leftarrow}(M) = \tau^{\leftarrow}(M')$

(v) $M[G\rangle_N M' \Rightarrow M \xrightarrow{\tau}{}^{*} \xrightarrow{G} \xrightarrow{\tau}{}^{*}_{FSI(N)} M'$

Proof (i): By $\forall s \in S.\ \tau^{\leftarrow}(s) = s$.

(ii): "$\Rightarrow$": $d(M) > 0 \Rightarrow \exists p \in M \exists t \in p^\bullet$. By construction of $FSI(N)$ then there exists a M' with $M[\{t_p\}\rangle M'$ as $\alpha(M)$ and hence $p_t \notin M$.

"$\Leftarrow$": $M \xrightarrow{\tau}_{FSI(N)} M' \Rightarrow \exists t_p \in U'.\ M[\{t_p\rangle M'$. And ${}^\circ t_p = \{p\}$ hence $p \in M$. By construction of $FSI(N)$ also $\exists t \in O.\ p \in {}^\bullet t$. Hence $d(M) = |M \cap \{s | s \in S, s^\bullet \neq \varnothing\}| \geq$ $|M \cap \{p\}| > 0$.

(iii): Consider any $t \in G \cap O$. Assume $(M \setminus {}^\circ t) \cap t^\circ \neq \varnothing$. Since $t^\circ \subseteq S$ let $p \in S$ such that $p \in M \cap t^\circ$. $p \notin {}^\bullet t$ as by construction of $FSI(N)$ also $p_t \in M$ and $\tau^{\leftarrow}(p) = p = \tau^{\leftarrow}(p_t)$ which would violate $\alpha(M)$. It follows that $(\tau^{\leftarrow}(M) \setminus {}^\bullet t) \cap t^\bullet \supseteq \{p\}$ and N would not be contact free as $\tau^{\leftarrow}(M) \in [M_0\rangle_N$ by $\alpha(M)$.

Consider any $t_p \in G \cap U'$. As $^\circ t_p = \{p\}$ and $t_p{}^\circ = \{p_t\}$ we have that $(M \setminus {}^\circ t) \cap t^\circ \neq \varnothing \Rightarrow p \in M \wedge p_t \in M$ but $\tau^\leftarrow(p) = p = \tau^\leftarrow(p_t)$ which would violate $\alpha(M)$.

$$\begin{aligned} M' &= (M \setminus \{s \mid s \in {}^\circ t, t \in G\}) \cup \{s \mid s \in t^\circ, t \in G\} \\ &= ((M \setminus \{s_t \mid s \in {}^\bullet t, t \in G \cap O\}) \setminus \{s \mid t_s \in G \cap U'\}) \cup \\ &\quad \{s_t \mid t_s \in G \cap U'\} \cup \{s \mid s \in t^\bullet, t \in G \cap O\} \end{aligned}$$

Therefore

$$\begin{aligned} \tau^\leftarrow(M') &= \tau^\leftarrow((M \setminus \{s_t \mid s \in {}^\bullet t, t \in G \cap O\}) \setminus \{s \mid t_s \in G \cap U'\}) \cup \\ &\quad \tau^\leftarrow(\{s_t \mid t_s \in G \cap U'\}) \cup \tau^\leftarrow(\{s \mid s \in t^\bullet, t \in G \cap O\}) \\ &= \tau^\leftarrow((M \setminus \{s_t \mid s \in {}^\bullet t, t \in G \cap O\}) \setminus \{s \mid t_s \in G \cap U'\}) \cup \\ &\quad \{s \mid t_s \in G \cap U'\} \cup \{s \mid s \in t^\bullet, t \in G \cap O\} \\ &= \tau^\leftarrow(M \setminus \{s_t \mid s \in {}^\bullet t, t \in G \cap O\}) \cup \\ &\quad \{s \mid t_s \in G \cap U'\} \cup \{s \mid s \in t^\bullet, t \in G \cap O\} \ . \end{aligned}$$

Take any $t \in G \cap O$ and any $s \in {}^\bullet t$. Then $s_t \in M$ and $\alpha(M) \Rightarrow s \notin M \wedge \nexists u \in O.\ u \neq t \wedge s_u \in M$. Hence $\tau^\leftarrow(M \setminus \{s_t \mid s \in {}^\bullet t, t \in G \cap O\}) = \tau^\leftarrow(M) \setminus \{s \mid s \in {}^\bullet t, t \in G \cap O\}$. Furthermore $\forall t_s \in G \cap U'.\ {}^\circ t_s = \{s\} \wedge s \in M$.

Thus we find

$$\tau^\leftarrow(M') = \tau^\leftarrow(M) \setminus \{s \mid s \in {}^\bullet t, t \in G \cap O\} \cup \{s \mid s \in t^\bullet, t \in G \cap O\} \ .$$

and conclude that $\tau^\leftarrow(M) \xrightarrow{G \cap O}_N \tau^\leftarrow(M')$.

We still need to prove that $\forall p, q \in M'.\ p \neq q \Rightarrow \tau^\leftarrow(p) \neq \tau^\leftarrow(q)$. Assume the contrary, i.e. there are $p, q \in M'$ with $p \neq q \wedge \tau^\leftarrow(p) = \tau^\leftarrow(q)$. Since $\alpha(M)$ at least one of p and q – say p – must not be present in M. Assume $p \in S^\tau$. Then there exist $s \in S, t \in O$ such that $s_t = p \wedge t_s \in G$ and thereby $\tau^\leftarrow(p) = s$. But then $s \in {}^\circ t_s \subseteq M$ and by $\alpha(M)$ there exists no $u \in O$ with $s_u \in M$. Since $t_s \in G \wedge s \notin t_s{}^\circ$ however $s \notin M'$. Furthermore by construction of *FSI*(N) $\forall v \in O \cup U'.\ s \in \tau^\leftarrow(v^\circ \cap S^\tau) \Rightarrow {}^\circ v = s$ and such a v could not fire with t_s in one step. Hence $\alpha(M')$ if $p \in S^\tau$.

If $p \in S$ then $\tau^\leftarrow(p) = p$ and by construction of *FSI*(N) there exists a $t \in G \cap O$ with $p \in t^\circ = t^\bullet$. However $\alpha(M) \Rightarrow M \in [M_0\rangle_N$ and $\tau^\leftarrow(M) \xrightarrow{G \cap O}_N \tau^\leftarrow(M') \Rightarrow {}^\bullet t \subseteq \tau^\leftarrow(M)$. Since N is contact free, then $(t^\bullet \setminus {}^\bullet t) \cap \tau^\leftarrow(M') = \varnothing$. Therefore $(\tau^\leftarrow(q) = \tau^\leftarrow(p) = p \wedge p \in t^\bullet \wedge q \in M') \Rightarrow \tau^\leftarrow(q) \in {}^\bullet t$. Furthermore by construction of *FSI*(N), ${}^\circ t \cap M' = \varnothing$ and it follows that either $q \in S \wedge q = p$ or $q = p_u$ for some $u \in O \wedge u \neq t$, which would violate $\alpha(M)$ since ${}^\circ t \subseteq M \Rightarrow p_t \in M$ and $\forall v \in O \cup U'.\ p_u \in v^\circ \Rightarrow {}^\circ v = \{p\}$. Hence $\alpha(M')$ if $p \in S$.

(iv): Let $t_s \in U'$ such that $M[\{t_s\}\rangle_{FSI(N)} M'$. Then, by construction of *FSI*(N), $s^\bullet \neq \varnothing$. Furthermore ${}^\circ t_s = \{s\} \wedge t_s{}^\circ = \{s_t\}$. Hence $M' = M \setminus \{s\} \cup \{s_t\}$ and $d(M') = d(M) - 1 \wedge \tau^\leftarrow(M') = \tau^\leftarrow(M)$.

(v): Assume $M[G\rangle_N M'$. $M \subseteq S$ by definition of N. Then, by construction of *FSI*(N), $M[\{t_s \mid t \in G, s \in {}^\bullet t\}\rangle_{FSI(N)}[\{t \mid t \in G\}\rangle_{FSI(N)} M'$. The first part of that execution can be split into a sequence of singletons. $\square$

After those basic properties are established, we can use them to prove more intuitive corollaries.

Lemma 3.2 Let N be a net.

> *FSI*(N) is divergence free.

Proof By Lemma 3.1 (i), (ii), (iii) and (iv). $\square$

Lemma 3.3 Let $N = (S, O, \varnothing, F, M_0)$ be a net.

> If N is contact free, so is *FSI*(N).

Proof By Lemma 3.1, (i) and (iii). $\square$

The following lemma states that the original net and the implementation can perform the same actions, provided that the final marking is an original marking. The correctness of this depends on the fact that all newly introduced deadlock situations will have some token "stuck" in a buffer place.

Lemma 3.4 Let $N = (S, O, \varnothing, F, M_0)$ be a net, $FSI(N) = (S \cup S^\tau, O, U', F', M_0)$ and $M_1 \in [M_0\rangle_N, M_2 \subseteq S$.

(i) $(M_1 \xrightarrow{G}_N M_2) \Leftrightarrow (M_1 \xrightarrow{\tau}{}^*_{FSI(N)} \xrightarrow{G}_{FSI(N)} \xrightarrow{\tau}{}^*_{FSI(N)} M_2)$

(ii) $(M_1 \overset{\sigma}{\Longrightarrow}_N M_2) \Leftrightarrow (M_1 \overset{\sigma}{\Longrightarrow}_{FSI(N)} M_2)$

Proof (i): "$\Rightarrow$": By applying Lemma 3.1 (v). "$\Leftarrow$": By using Lemma 3.1 (i), (v) and (iii) we find that $\alpha(M_1) \wedge \alpha(M_2)$. The result then follows from Lemma 3.1 (iii), as $\tau^{\leftarrow}(M_1) = M_1$ and $\tau^{\leftarrow}(M_2) = M_2$ since both $M_1 \subseteq S \wedge M_2 \subseteq S$.

(ii): By complete induction on the length of σ. For $\sigma = \varepsilon$ "$\Rightarrow$" is trivially true and "$\Leftarrow$" also holds because $\forall t \in U'.\ t^\bullet \cap S^\tau \neq \varnothing$ and therefore $\forall M_2' \subseteq S.\ M_1 \overset{\varepsilon}{\Longrightarrow}_{FSI(N)} M_2' \Rightarrow M_2' = M_1$.

Let $t \in O$. If (ii) holds for some σ then it also holds for σt due to (i) with $G = \{t\}$. $\square$

In addition to the above lemma it is also the case that the implementation can always simulate the original "optimally" in the sense that no superfluous transitions are fired and every marking which existed in the original trace is also reached by the implementation.

Lemma 3.5 Let $N = (S, O, \varnothing, F, M_0)$ be a net and let $FSI(N) = (S \cup S^\tau, O, U', F', M_0)$.

Let $M \subseteq S \cup S^\tau, \sigma \in O^*$ such that $M_0 \overset{\sigma}{\Longrightarrow}_{FSI(N)} M$ and let $M_S := \tau^{\leftarrow}(M)$.

Then $M_0 \overset{\sigma}{\Longrightarrow}_{FSI(N)} M_S$ and $\nexists M'_S \subseteq S.\ M'_S \neq M_S \wedge M_0 \overset{\sigma}{\Longrightarrow}_{FSI(N)} M'_S$.

Proof By induction over the length of σ using Lemma 3.1 (i) and (iii) $M_0 \overset{\sigma}{\Longrightarrow}_N \tau^{\leftarrow}(M)$ wherefrom by using Lemma 3.1 (v) also $M_0 \overset{\sigma}{\Longrightarrow}_{FSI(N)} \tau^{\leftarrow}(M)$.

Assume any other $M' \subseteq S$ exists such that $M_0 \overset{\sigma}{\Longrightarrow}_{FSI(N)} M'$. Then $M_0 \overset{\sigma}{\Longrightarrow}_N \tau^{\leftarrow}(M')$. But by Lemma 2.1 then $\tau^{\leftarrow}(M') = \tau^{\leftarrow}(M)$. Since $M \subseteq S \wedge M' \subseteq S$ then $M = M'$. $\square$

All those lemmas above can be combined to the already mentioned fact that the only difference in behaviour between the original net and its implementation is the introduction of new deadlocks, which formally result in additional failures.

Proposition 3.1 Let $N = (S, O, \varnothing, F, M_0)$ be a net.

Then $\mathscr{F}(N) \subseteq \mathscr{F}(FSI(N))$.

Proof Let $FSI(N) = (S \cup S^\tau, O, U', F', M_0)$. Let $<\sigma, X> \in \mathscr{F}(N)$, $t \in X$ and let $M_1 \subseteq S$ such that $M_0 \overset{\sigma}{\Longrightarrow}_N M_1$.

By using Lemma 3.1 (i), (iii) and (v) in an induction over σ, also $M_0 \overset{\sigma}{\Longrightarrow}_{FSI(N)} M_1$. Using Lemma 3.1 (iv) and (ii) there exists a marking M'_1 such that $M'_1 \overset{\tau}{\not\longrightarrow}_{FSI(N)} \wedge \tau^{\leftarrow}(M'_1) = \tau^{\leftarrow}(M_1) \wedge \alpha(M'_1)$.

Consider a transition $t \in X$. Assume that t is not refused in M'_1 by $FSI(N)$, that is $\exists M'_2 \subseteq S \cup S^\tau.\ M'_1[\{t\}\rangle_{FSI(N)} M'_2$.

Then by Lemma 3.1 (iii) and $\tau^{\leftarrow}(M'_1) = M_1$ immediately $M_1 \overset{\{t\}}{\longrightarrow} \tau^{\leftarrow}(M_2)$ which is a contradiction. Therefore $<\sigma, X> \in FSI(N)$. $\square$

Finally we define the class of nets which are asynchronous, by testing whether if they where to be implemented asynchronously they would still function correctly. Actually we define multiple classes as different equivalence relations lead to different results.

Definition 3.4

The class of *fully symmetrically asynchronous nets respecting linear time equivalence* is defined as $FSA(L) := \{N \mid FSI(N) \simeq_{CPT} N\}$.

The class of *fully symmetrically asynchronous nets respecting branching time equivalence* is defined as $FSA(B) := \{N \mid FSI(N) \simeq_F N\}$.

We also have obtained the following semi-structural characterisation of $FSA(B)$.

Definition 3.5
A net $N = (S, O, \varnothing, F, M_0)$ *has a partially reachable conflict* iff $\exists t, u \in O.\ t \neq u \wedge {}^\bullet t \cap {}^\bullet u \neq \varnothing$ and $\exists M \in [M_0\rangle.\ {}^\bullet t \subseteq M \vee {}^\bullet u \subseteq M$.

The correctness of the characterisation is proven below.

Theorem 3.1
A net $N = (S, O, \varnothing, F, M_0)$ is in $FSA(B)$ iff N has no partially reachable conflict.

Proof Let $FSI(N) = (S \cup S^\tau, O, U', F', M_0)$.

"$\Rightarrow$": Assume N has a partially reachable conflict. Then there exist $t, u \in O$, $t \neq u$, $\sigma \in O^*$ and $M \subseteq S$ such that $M_0 \stackrel{\sigma}{\Longrightarrow}_N M$, ${}^\bullet t \cap {}^\bullet u \neq \varnothing$ and ${}^\bullet t \subseteq M \vee {}^\bullet u \subseteq M$. Without loss of generality assume that ${}^\bullet t \subseteq M$.

For every $<\sigma, X> \in \mathscr{F}(N)$ we then know that $t \notin X$ by Lemma 2.1.

However $M_0 \stackrel{\sigma}{\Longrightarrow}_{FSI(N)} M$ by Lemma 3.4. Let $p \in {}^\bullet t \cap {}^\bullet u$. Then, by construction of $FSI(N)$, there exists $M_1 \subseteq S \cup S^\tau$ with $M[\{u_p\}\rangle M_1$, $p \notin M_1$ and since $t \neq u$ also $p_t \notin M_1$. Now let $M_2 \subseteq S \cup S^\tau$ such that $M_1 \stackrel{\tau}{\longrightarrow}^*_{FSI(N)} M_2 \wedge M_2 \stackrel{\tau}{\not\longrightarrow}^*_{FSI(N)}$ (which exists according to Lemma 3.2). Since $\forall v \in U'.\ p \notin v^\bullet \wedge (p_t \in v^\bullet \Rightarrow p \in {}^\bullet v)$ we know that $p_t \notin M_2$.

But then $M_2 \stackrel{\{t\}}{\not\longrightarrow}$ and there exists a failure pair $<\sigma, \{t\}> \in \mathscr{F}(FSI(N))$. We thereby know that $\mathscr{F}(FSI(N)) \neq \mathscr{F}(N)$.

"$\Leftarrow$": Assume $N \notin FSA(B)$. Then $\mathscr{F}(FSI(N)) \neq \mathscr{F}(N)$ and $\mathscr{F}(FSI(N)) \setminus \mathscr{F}(N) \neq \varnothing$ by Proposition 3.1.

Let $<\sigma, X> \in \mathscr{F}(FSI(N)) \setminus \mathscr{F}(N)$. Then there exists an $M_1 \subseteq S \cup S^\tau$ such that $M_0 \stackrel{\sigma}{\Longrightarrow}_{FSI(N)} M_1 \wedge M_1 \stackrel{\tau}{\not\longrightarrow} \wedge \forall t \in X.\ M_1 \stackrel{\{t\}}{\not\longrightarrow}$.

By Lemma 3.5 then also $M_0 \stackrel{\sigma}{\Longrightarrow}_{FSI(N)} \tau^\leftarrow(M_1)$ and by Lemma 3.4 $M_0 \stackrel{\sigma}{\Longrightarrow}_N \tau^\leftarrow(M_1)$.

Let $t \in X$ such that $\tau^\leftarrow(M_1) \stackrel{t}{\Longrightarrow}_N$ (which exists, otherwise $<\sigma, X> \in \mathscr{F}(N)$). Let $p \in {}^\bullet t$ such that $p_t \notin M_1$ (such p_t exists, otherwise $M_1 \stackrel{\{t\}}{\longrightarrow}_{FSI(N)}$).

Since $\tau^\leftarrow(M_1) \stackrel{t}{\Longrightarrow}_N$ it follows that $p \in \tau^\leftarrow(M_1)$. But $p \notin M_1$ otherwise $M_1 \stackrel{\tau}{\longrightarrow}_{FSI(N)}$ which would be a contradiction. Hence there must exists some $u \in O$ with $p_u \in M_1$. By construction of $FSI(N)$ then $p \in {}^\bullet u$.

But then $t, u \in O \wedge {}^\bullet t \cap {}^\bullet u \neq \varnothing \wedge \tau^\leftarrow(M_1) \in [M_0\rangle_N \wedge {}^\bullet t \subseteq \tau^\leftarrow(M_1)$ and N has a partially reachable conflict. $\square$

From those results it is already visible that when considering branching time equivalences only very simple nets are failures equivalent to their fully symmetrically asynchronous implementation. An example which already fails is shown in Figure 3.1.

The net class $FSA(L)$ is substantially larger. However it coincides with some other class of nets which will be defined in the next section. The proof of that coincidence must naturally go after the definitions of the other net class and is contained in the next section.

4 Symmetric Asynchrony

Since we are interested in more substantial results regarding branching time equivalences, we change our definition somewhat and only insert invisible transitions wherever a transition has multiple preplaces, when the synchronous removal of tokens is really essential.

Definition 4.1 Let $N = (S, O, \varnothing, F, M_0)$ be a net. Let $O^b = \{t \mid t \in O, |{}^\bullet t| > 1\}$.
The *symmetrically asynchronous implementation* of N is defined as the net $\mathit{SI}(N) := (S \cup S^\tau, O, U', F', M_0)$ with

$$\begin{aligned}
S^\tau &:= \{s_t \mid t \in O^b, s \in {}^\bullet t\}\ , \\
U' &:= \{t_s \mid t \in O^b, s \in {}^\bullet t\} \text{ and} \\
F' &:= F \cap \Big((O \times S) \cup (S \times (O \setminus O^b))\Big) \\
&\quad\ \cup \{(s, t_s) \ \mid t \in O^b, s \in {}^\bullet t\} \\
&\quad\ \cup \{(t_s, s_t) \mid t \in O^b, s \in {}^\bullet t\} \\
&\quad\ \cup \{(s_t, t) \ \mid t \in O^b, s \in {}^\bullet t\}\ .
\end{aligned}$$

The effect of this transformation can be seen in Figure 1.1. A discussion in what sense this new transformation is consistent with intuition follows later, after the details are made more clear.

Similar to Section 3, we use ${}^\circ x$ and x° if the flow relation of the implementation is described. As before we establish basic properties of our transformation which will be useful later on. To do so, we again wish to undo the effect of extraneous τ-transitions. The function to do so is the same $\tau^\leftarrow$ defined earlier.

It turns out that the basic principles of Lemma Lemma 3.1 also holds for this modified version of asynchronous implementation. However the invariant and distance functions need slight modification.

Definition 4.2 Let $N = (S, O, \varnothing, F, M_0)$ be a net and $\mathit{SI}(N) = (S \cup S^\tau, O, U', F', M_0)$.
The predicate $\beta \subseteq \mathcal{P}(S \cup S^\tau)$ is defined as $\beta(M) :\Leftrightarrow \tau^\leftarrow(M) \in [M_0\rangle_N \wedge \forall p, q \in M.\ \tau^\leftarrow(p) \neq \tau^\leftarrow(q)$. The function $e : \mathcal{P}(S \cup S^\tau) \rightarrow \mathbb{N}$ is defined as $e(M) := |M \cap \{s \mid s \in S, \exists t \in s^\bullet.\ |{}^\bullet t| > 1\}|$.

Lemma 4.1 Let $N = (S, O, \varnothing, F, M_0)$ be a net, $\mathit{SI}(N) = S \cup S^\tau, O, U', F', M_0)$ and $M \subseteq S \cup S^\tau$.

(i) $\beta(M_0)$

(ii) $\beta(M) \Rightarrow (e(M) > 0 \Leftrightarrow \exists M' \subseteq S \cup S^\tau.\ M \xrightarrow{\tau}_{\mathit{SI}(N)} M')$

(iii) $M[G\rangle_{\mathit{SI}(N)} M' \wedge \beta(M) \Rightarrow \forall t \in G.\ (M \setminus {}^\circ t) \cap t^\circ = \varnothing \wedge \tau^{\leftarrow}(M) \xrightarrow{G \cap O}_N \tau^{\leftarrow}(M') \wedge \beta(M')$

(iv) $M \xrightarrow{\tau}_{\mathit{SI}(N)} M' \Rightarrow e(M) > e(M') \wedge \tau^{\leftarrow}(M) = \tau^{\leftarrow}(M')$

(v) $M[G\rangle_N M' \Rightarrow M \xrightarrow{\tau}{}^{*} \xrightarrow{G} \xrightarrow{\tau}{}^{*}_{\mathit{SI}(N)} M'$

Proof (i): By $\forall s \in S.\ \tau^{\leftarrow}(s) = s$.

(ii): "$\Rightarrow$": $e(M) > 0 \Rightarrow \exists p \in M \exists t \in p^\bullet.\ |{}^\bullet t| > 1$. By construction of $\mathit{SI}(N)$ then there exists a M' with $M[\{t_p\}\rangle M'$ as $\beta(M)$ and hence $p_t \notin M$.

"$\Leftarrow$": $M \xrightarrow{\tau}_{\mathit{SI}(N)} M' \Rightarrow \exists t_p \in U'.\ M[\{t_p\rangle M'$. And ${}^\circ t_p = \{p\}$ hence $p \in M$. By construction of $\mathit{SI}(N)$ also $\exists t \in O.\ p \in {}^\bullet t \wedge |{}^\bullet t| > 1$. Hence $e(M) = |M \cap \{s | s \in S, \exists t \in s^\bullet.\ |{}^\bullet t| > 1\}| \geq |M \cap \{p\}| > 0$.

(iii): Consider any $t \in G \cap O$. Assume $(M \setminus {}^\circ t) \cap t^\circ \neq \varnothing$. Since $t^\circ \subseteq S$ let $p \in S$ such that $p \in (M \setminus {}^\circ t) \cap t^\circ$.

There are two cases:

$|{}^\bullet t| = 1$: If $p \in {}^\bullet t$ also $p \in {}^\circ t$ wich would be a contradiction with $p \in M \setminus {}^\circ t$. Otherwise $p \notin {}^\bullet t$ and $(\tau^{\leftarrow}(M) \setminus {}^\bullet t) \cap t^\bullet \supseteq \{p\}$ and N would not be contact free as $\tau^{\leftarrow}(M) \in [M_0\rangle_N$ by $\beta(M)$.

$|{}^\bullet t| > 1$: $p \notin {}^\bullet t$ as by construction of $\mathit{SI}(N)$ also $p_t \in M$ and $\tau^{\leftarrow}(p) = p = \tau^{\leftarrow}(p_t)$ which would violate $\beta(M)$. Hence $(\tau^{\leftarrow}(M) \setminus {}^\bullet t) \cap t^\bullet \supseteq \{p\}$ and N would not be contact free as $\tau^{\leftarrow}(M) \in [M_0\rangle_N$ by $\beta(M)$.

Consider any $t_p \in G \cap U'$. As ${}^\circ t_p = \{p\}$ and ${t_p}^\circ = \{p_t\}$ we have that $(M \setminus {}^\circ t) \cap t^\circ \neq \varnothing \Rightarrow p \in M \wedge p_t \in M$ but $\tau^{\leftarrow}(p) = p = \tau^{\leftarrow}(p_t)$ which would violate $\beta(M)$.

Let $O^b := \{t \mid t \in O, |{}^\bullet t| > 1\}$ and $O^{nb} := \{t \mid t \in O, |{}^\bullet t| = 1\}$.

$$\begin{aligned} M' &= (M \setminus \{s \mid s \in {}^\circ t, t \in G\}) \cup \{s \mid s \in t^\circ, t \in G\} \\ &= (M \setminus (\{s_t \mid s \in {}^\bullet t, t \in G \cap O^b\} \cup \{s \mid s \in {}^\bullet t, t \in G \cap O^{nb}\} \cup \{s \mid t_s \in G \cap U'\})) \cup \\ &\quad \{s_t \mid t_s \in G \cap U'\} \cup \{s \mid s \in t^\bullet, t \in G \cap O\} \end{aligned}$$

Therefore

$$\begin{aligned}\tau^{\leftarrow}(M') = \tau^{\leftarrow}(M \setminus (\{s_t | s \in {}^\bullet t, t \in G \cap O^b\} \cup \{s | s \in {}^\bullet t, t \in G \cap O^{nb}\} \cup \{s | t_s \in G \cap U'\})) \cup \\ \tau^{\leftarrow}(\{s_t \mid t_s \in G \cap U'\}) \cup \tau^{\leftarrow}(\{s \mid s \in t^\bullet, t \in G \cap O\}) \\ = \tau^{\leftarrow}(M \setminus (\{s_t | s \in {}^\bullet t, t \in G \cap O^b\} \cup \{s | s \in {}^\bullet t, t \in G \cap O^{nb}\} \cup \{s | t_s \in G \cap U'\})) \cup \\ \{s \mid t_s \in G \cap U'\} \cup \{s \mid s \in t^\bullet, t \in G \cap O\} \\ = \tau^{\leftarrow}(M \setminus (\{s_t \mid s \in {}^\bullet t, t \in G \cap O^b\} \cup \{s \mid s \in {}^\bullet t, t \in G \cap O^{nb}\})) \cup \\ \{s \mid t_s \in G \cap U'\} \cup \{s \mid s \in t^\bullet, t \in G \cap O\} \; .\end{aligned}$$

Take any $t \in G \cap O^b$ and any $s \in {}^\bullet t$. Then $s_t \in M$ and $\beta(M) \Rightarrow s \notin M \wedge \nexists u \in O.\ u \neq t \wedge s_u \in M$. Take any $t \in G \cap O^{nb}$ and the $s \in {}^\bullet t$. Then $s \in M$ and $\beta(M) \Rightarrow \nexists u \in O.\ s_u \in M$.

Hence

$$\begin{aligned}\tau^{\leftarrow}(M \setminus (\{s_t \mid s \in {}^\bullet t, t \in G \cap O^b\} \cup \{s \mid s \in {}^\bullet t, t \in G \cap O^{nb}\})) \\ = \tau^{\leftarrow}(M) \setminus (\{s \mid s \in {}^\bullet t, t \in G \cap O^b\} \cup \{s \mid s \in {}^\bullet t, t \in G \cap O^{nb}\}) \\ = \tau^{\leftarrow}(M) \setminus \{s \mid s \in {}^\bullet t, t \in G \cap O\} \; .\end{aligned}$$

Furthermore $\forall t_s \in G \cap U'.\ {}^\circ t_s = \{s\} \wedge s \in M$.

Thus we find

$$\tau^{\leftarrow}(M') = \tau^{\leftarrow}(M) \setminus \{s \mid s \in {}^\bullet t, t \in G \cap O\} \cup \{s \mid s \in t^\bullet, t \in G \cap O\} \; .$$

and conclude that $\tau^{\leftarrow}(M) \xrightarrow{G \cap O}_N \tau^{\leftarrow}(M')$.

We still need to prove that $\forall p, q \in M'.\ p \neq q \Rightarrow \tau^{\leftarrow}(p) \neq \tau^{\leftarrow}(q)$. Assume the contrary, i.e. there are $p, q \in M'$ with $p \neq q \wedge \tau^{\leftarrow}(p) = \tau^{\leftarrow}(q)$. Since $\beta(M)$ at least one of p and q – say p – must not be present in M. Assume $p \in S^\tau$. Then there exist $s \in S, t \in O$ such that $s_t = p \wedge t_s \in G$ and thereby $\tau^{\leftarrow}(p) = s$. But then $s \in {}^\circ t_s \subseteq M$ and by $\beta(M)$ there exists no $u \in O$ with $s_u \in M$. Since $t_s \in G \wedge s \notin t_s{}^\circ$ however $s \notin M'$. Furthermore by construction of $SI(N)$ $\forall v \in O \cup U'.\ s \in \tau^{\leftarrow}(v^\circ \cap S^\tau) \Rightarrow {}^\circ v = s$ and such a v could not fire with t_s in one step. Hence $\beta(M')$ if $p \in S^\tau$.

If $p \in S$ then $\tau^{\leftarrow}(p) = p$ and by construction of $SI(N)$ there exists a $t \in G \cap O$ with $p \in t^\circ = t^\bullet$. However $\beta(M) \Rightarrow M \in [M_0\rangle_N$ and $\tau^{\leftarrow}(M) \xrightarrow{G \cap O}_N \tau^{\leftarrow}(M') \Rightarrow {}^\bullet t \subseteq \tau^{\leftarrow}(M)$. Since N is contact free, then $(t^\bullet \setminus {}^\bullet t) \cap \tau^{\leftarrow}(M') = \varnothing$. Therefore $(\tau^{\leftarrow}(q) = \tau^{\leftarrow}(p) = p \wedge p \in t^\bullet \wedge q \in M') \Rightarrow \tau^{\leftarrow}(q) \in {}^\bullet t$.

If $|{}^\bullet t| > 1$ by construction of $SI(N)$, ${}^\circ t \cap M' = \varnothing$ and it follows that either $q \in S \wedge q = p$ or $q = p_u$ for some $u \in O \wedge u \neq t$, which would violate $\beta(M)$ since ${}^\circ t \subseteq M \Rightarrow p_t \in M$ and $\forall v \in O \cup U'.\ p_u \in v^\circ \Rightarrow {}^\circ v = \{p\}$. Hence $\beta(M')$ if $p \in S$. Otherwise ${}^\bullet t = \{\tau^{\leftarrow}(q)\} = \{p\}$ and also ${}^\circ t = \{p\}$. But p was assumed to be not in M.

Hence $\beta(M')$ if $p \in S$.

(iv): Let $t_s \in U'$ such that $M[\{t_s\}\rangle_{SI(N)} M'$. Then, by construction of $SI(N)$, $t \in s^\bullet \wedge |{}^\bullet t| > 1$. Furthermore ${}^\circ t_s = \{s\} \wedge t_s{}^\circ = \{s_t\}$. It follows that $M' = M \setminus \{s\} \cup \{s_t\}$ and $e(M') = e(M) - 1 \wedge \tau^\leftarrow(M') = \tau^\leftarrow(M)$.

(v): Assume $M[G\rangle_N M'$. $M \subseteq S$ by definition of N. Let $O^b := \{t \mid t \in O, |{}^\bullet t| > 1\}$. Then, by construction of $SI(N)$, $M[\{t_s \mid t \in G \cap O^b, s \in {}^\bullet t\}\rangle_{SI(N)}[\{t \mid t \in G\}\rangle_{SI(N)} M'$. The first part of that execution can be split into a sequence of singletons. □

As Lemma 4.1 is basically the same as Lemma 3.1 it should come as no surprise that the corollaries also hold.

Lemma 4.2 Let N be a net.

$SI(N)$ is divergence free.

Proof By Lemma 4.1 (i), (ii), (iii) and (iv). □

Lemma 4.3 Let $N = (S, O, \varnothing, F, M_0)$ be a net.

If N is contact free, so is $SI(N)$.

Proof By Lemma 4.1, (i) and (iii). □

The following lemma shows that all behaviours of N can be simulated by $SI(N)$ and vice versa for visible behaviours of $SI(N)$. Note however that $SI(N)$ might be able to deadlock in more cases than N. One typical case is shown in Figure 4.1.

Lemma 4.4 Let $N = (S, O, \varnothing, F, M_0)$ be a net, $SI(N) = (S \cup S^\tau, O, U', F', M_0)$ and $M_1 \in [M_0\rangle_N, M_2 \subseteq S$.

(i) $(M_1 \xrightarrow{G}_N M_2) \Leftrightarrow (M_1 \xrightarrow{\tau}{}^*_{SI(N)} \xrightarrow{G}_{SI(N)} \xrightarrow{\tau}{}^*_{SI(N)} M_2)$

(ii) $(M_1 \overset{\sigma}{\Longrightarrow}_N M_2) \Leftrightarrow (M_1 \overset{\sigma}{\Longrightarrow}_{SI(N)} M_2)$

Proof Completely parallel to Lemma 3.4, using Lemma 4.1 instead of Lemma 3.1. □

Lemma 4.5 Let $N = (S, O, \varnothing, F, M_0)$ be a net and let $SI(N) = (S \cup S^\tau, O, U', F', M_0)$.

Let $M \subseteq S \cup S^\tau, \sigma \in O^*$ such that $M_0 \overset{\sigma}{\Longrightarrow}_{SI(N)} M$ and let $M_S := \tau^\leftarrow(M)$.

Then $M_0 \overset{\sigma}{\Longrightarrow}_{SI(N)} M_S$ and $\nexists M'_S \subseteq S.\ M'_S \neq M_S \wedge M_0 \overset{\sigma}{\Longrightarrow}_{SI(N)} M'_S$.

Proof Completely parallel to Lemma 3.5, using Lemma 4.1 instead of Lemma 3.1. □

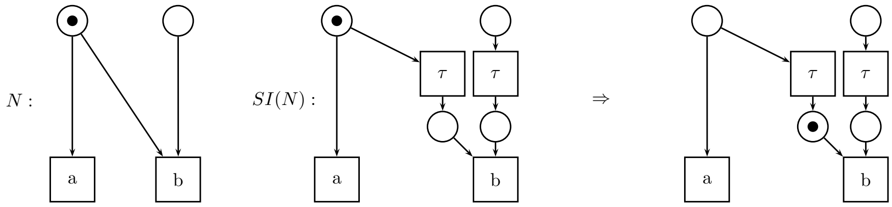

Figure 4.1: The implementation reached a deadlock, which was not possible before, hence $N \notin SA(B)$. But $N \in AA(H, B)$.

Proposition 4.1 Let $N = (S, O, \varnothing, F, M_0)$ be a net and $SI(N) = (S \cup S^\tau, O, U', F', M_0)$.

Then $\mathscr{F}(N) \subseteq \mathscr{F}(SI(N))$.

Proof Completely parallel to Proposition 3.1, using Lemma 4.1 instead of Lemma 3.1. □

Similar as we did for interleaving behaviour in Lemma 4.4, we also relate the possible processes of a net to those of its implementation. Due to time constraints the proofs and basic properties cannot be presented as elaborate as we did for the interleaving case.

Lemma 4.6 Let $N = (S, O, \varnothing, F, M_0)$ be a net, $N_f = (S_f, O_f, \varnothing, F_f, M_{0f})$ an occurrence net and $f : S_f \cup O_f \to S \cup O$ a function. Let $SI(N) = (S \cup S^\tau, O, U', F', M_0)$ and $SI(N_f) = (S_f \cup S_f^\tau, O_f, U_f', F_f', M_{0f})$.

Let $SI(f) : S_f \cup S_f^\tau \cup O_f \cup U_f' \to S \cup S^\tau \cup O \cup U$ be the function defined by

$$SI(f)(x) := \begin{cases} s_t & \text{iff } x = p_u \in S_f^\tau \text{ with } s = f(p) \text{ and } t = f(u) \\ t_s & \text{iff } x = u_p \in U_f' \text{ with } s = f(p) \text{ and } t = f(u) \\ f(x) & \text{otherwise} \end{cases} .$$

Then f is a process of N iff $SI(f)$ is a process of $SI(N)$.

Proof "⇒": Assume f is a process of N. We show that $SI(f)$ is a process of $SI(N)$.

$$\begin{aligned} SI(f)(S_f) &= f(S_f) \subseteq S \\ SI(f)(S_f^\tau) &= \{s_t \mid p_u \in S_f^\tau, s = f(p), t = f(u)\} \subseteq S \cup S^\tau \\ SI(f)(O_f) &= f(O_f) \subseteq O \\ SI(f)(U_f') &= \{t_s \mid u_p \in U_f', s = f(p), t = f(u)\} \subseteq U' \\ SI(f)(M_{0f}) &= f(M_{0f}) = M_0 \end{aligned}$$

We show that $SI(f)$ is injective over slices.

Let C be a slice of $SI(N_f)$, then $\tau^{\leftarrow}(C)$ is a slice of N_f by construction of $SI(N_f)$. Let $x, y \in C$ such that $SI(f)(x) = SI(f)(y)$.

If $x \in S_f$ then by construction of $SI(f)$ also $y \in S_f$, $f(x) = SI(f)(x) = SI(f)(y) = f(y)$ and, since f is injective over slices, $x = y$.

Else $x \in S_f^\tau$ and by construction of $SI(f)$ also $y \in S_f^\tau$. Therefore let $p_u = x, q_v = y$ and $s_t = SI(f)(x) = SI(f)(y)$. Then $f(u) = f(v) = t \wedge f(p) = f(q) = s$ and thereby $p = q$ as f is injective over slices and $p = \tau^{\leftarrow}(p_u) \wedge q = \tau^{\leftarrow}(q_v)$. Yet $u \in p^\bullet$ and $v \in p^\bullet$ but $|p^\bullet| = 1$ and hence $u = v$. Therefore $p_u = q_v$.

It remains to be shown that $SI(f)$ respects pre- and postsets of transitions.

Let $t \in O_f \cup U'_f$.

$SI(f)$ respects the postset of t: If $t \in O_f$, then $SI(f)(t)^\bullet = f(t)^\bullet = f(t^\bullet) = SI(t)(t^\bullet)$ as $t^\bullet \subseteq S_f$. Else $t = u_p \in U'_f$ and $SI(f)(t)^\bullet = \left(f(u)_{f(p)}\right)^\bullet = \{f(p)_{f(u)}\} = SI(f)(\{p_u\}) = SI(f)(t^\bullet)$.

$SI(f)$ respects the preset of t: $t \in O_f \setminus O_f^b$ or $t \in O_f^b$ or $t \in U'_f$. If $t \in O_f \setminus O_f^b$, then ${}^\bullet SI(f)(t) = {}^\bullet f(t) = f({}^\bullet t) = SI(f)({}^\bullet t)$ as ${}^\bullet t \subseteq S_f$. If $t \in O_f^b$ then ${}^\bullet SI(f)(t) = {}^\bullet f(t) = \{f(s)_{f(t)} \mid (f(s), f(t)) \in F\} = SI(f)({}^\bullet t)$. If $t = u_p \in U'_f$ then ${}^\bullet SI(f)(t) = {}^\bullet \left(f(u)_{f(p)}\right) = \{f(p)\} = SI(f)(\{p\}) = SI(f)({}^\bullet t)$.

"$\Leftarrow$": Assume $SI(f)$ is a process of $SI(N)$. We show that f is a process of N.

$$\begin{aligned} f(S_f) &= SI(f)(S_f) \subseteq S \cup S^\tau \wedge f(S_f) \subseteq S \cup O \text{ hence } f(S_f) \subseteq S \\ f(O_f) &= SI(f)(O_f) \subseteq O \\ f(M_{0f}) &= SI(f)(M_{0f}) = M_0 \end{aligned}$$

We show that f is injective over slices.

Let C be a slice of N_f, then C is also a slice of $SI(N_f)$ by construction of $SI(N_f)$. Let $x, y \in C$ such that $f(x) = f(y)$. Then also $SI(f)(x) = SI(f)(y)$ since $x, y \in S_f$. Since $SI(f)$ is injective over slices, $x = y$.

It remains to be shown that f respects pre- and postsets of transitions.

Let $t \in O_f$.

f respects the postset of t: $f(t^\bullet) = SI(f)(t^\bullet) = SI(f)(t)^\bullet = f(t)^\bullet$ as $t^\bullet \subseteq S_f$.

f respects the preset of t:

Let $O^b := \{t \in O \mid 1 < |{}^\bullet t|\}$ and $O_f^b := \{t \in O_f \mid 1 < |{}^\bullet t|\}$.

If $t \notin O_f^b$ then $f({}^\bullet t) = SI(f)({}^\bullet t) = {}^\bullet SI(f)(t) = {}^\bullet f(t)$ as $\{s \mid (s,t) \in F'_f\} \subseteq S_f$.

Else $t \in O_f^b$. Let ${}^\star x = \{y \mid (y, x) \in F'_f\}$ and $x^\star = \{y \mid (x, y) \in F'_f\}$. By construction of $SI(f)$, $\forall v \in O^b.\ {}^\bullet v = {}^{\circ\circ\circ} v$, $\forall u \in O_f^b.\ {}^\bullet u = {}^{\star\star\star} u$ and similarly for the postsets. Since $SI(f)$ is a process, $\forall u \in O_f^b.\ SI(f)({}^\star u) = {}^\circ SI(f)(u)$ and similarly for the postset.

Thereby

$$f({}^\bullet t) = SI(f)({}^\bullet t) = SI(f)({}^{\star\star\star}t) = {}^\circ SI(f)({}^{\star\star}t)) = {}^\circ SI(f)(\{u \in O_f \cup U'_f \mid u^\star \cap {}^\star t \neq \varnothing\}) \ .$$

By construction of $SI(N)$, there exists exactly one $u \in O_f \cup U'_f$ for each $p_t \in {}^\star t$ such that $p_t \in u^\star$, namely $u = t_p$. Furthermore there exists exactly one $v \in O \cup U'$ with $SI(f)(p_t) \in v^\circ$, namely $v = f(t)_{f(p)} = SI(f)(u)$. Hence $\forall p_t \in {}^\star t.\ SI(f)({}^\star p_t) = {}^\circ SI(f)(p_t)$.

Thereby we have ${}^\circ SI(f)({}^{\star\star}t) = {}^{\circ\circ} SI(f)({}^\star t) = {}^{\circ\circ\circ} SI(f)(t) = {}^\bullet SI(f)(t) = {}^\bullet f(t)$. □

$SI(f)$ will be used later with the same definition as in Lemma 4.6.

Not only can every visible behaviour of a net be simulated by its implementation, but the only difference between the sets of possible behaviours is the existence of new possible deadlocks in the transformed version of the net.

Lemma 4.7 Let $N = (S, O, \varnothing, F, M_0)$ be a net.

Then $MVP(N) \subseteq MVP(SI(N))$.

Proof Let f be a maximal process of N. Let $N_f = (S_f, O_f, \varnothing, F_f, M_{0f})$ be the occurrence net f is based on. Then $SI(f)$ is a process of $SI(N)$ according to Lemma 4.6 based upon some occurrence net $SI(N_f) = (S_f \cup S_f^\tau, O_f, U_f, F'_f, M_{0f})$.

We show that $SI(f)$ can be extended to a maximal process of $SI(N)$ without changing the visible pomset.

Note that $SI(N_f)^\circ = N_f^\circ$ by construction of $SI(N_f)$. Assume there exists a $t \in O_f$ such that $SI(N_f)^\circ \overset{t}{\Longrightarrow}_{SI(N)}$.

If $|{}^\bullet t| = 1$ then ${}^\bullet t \subseteq N_f^\circ$ because $\forall u \in U_f.\ u^\bullet \subseteq S_f^\tau$ but ${}^\bullet t \subseteq S_f$. Thus f would not be maximal.

In the case of $|{}^\bullet t| > 1$, consider a place $p_t \in {}^\bullet t$. Since $SI(N_f)^\circ \subseteq S_f$ and by construction of $SI(N_f)$ it follows that $SI(N_f)^\circ \xrightarrow{\tau}{}^* \xrightarrow{\{t_p\}} \xrightarrow{\tau}{}^* \xrightarrow{\{t\}}$. But ${}^\bullet t_p = \{p\}$. As above $\forall u \in U_f.\ u^\bullet \subseteq S_f^\tau$. Hence the invisible transitions in the first $\xrightarrow{\tau}{}^*$ cannot have marked p and thus $p \in SI(N_f)^\circ$. Repeating this argument for each $p_t \in {}^\bullet t$ we find that $\{s \mid (s,t) \in F\} \subseteq N_f^\circ$. Thus f would not be maximal.

Therefore no visible transition can subsequently get enabled in $SI(N_f)$ if f is maximal. Furthermore $\forall t \in U_f.\ {}^\bullet t \subseteq S_f \wedge t^\bullet \subseteq S_f^\tau$ and hence only finitely many invisible transitions are possible. Thus $SI(f)$ can be extended to a maximal process of $SI(N)$ with $VP(SI(f)) = VP(f)$. □

By observing which nets preserve their behaviour if implemented asynchronously, we can classify them as follows.

Definition 4.3

(i) The class of *symmetrically asynchronous nets respecting linear time equivalence* is defined as $SA(L) := \{N \mid SI(N) \simeq_{CPT} N\}$.

(ii) The class of *symmetrically asynchronous nets respecting branching time equivalence* is defined as $SA(B) := \{N \mid SI(N) \simeq_{F} N\}$.

We return to the question of how large $FSA(L)$ is. It turns out that $FSA(L) = SA(L)$ as shown in the following lemma. We only prove one direction, as the other is intuitively clear. Requiring more asynchrony should not enlarge a class of nets.

Proposition 4.2 Let N be a net with $SI(N) \simeq_{CPT} N$.

Then $FSI(N) \simeq_{CPT} N$.

Proof Let $SI(N) = (S \cup S^{\tau}, O, U', F', M_0)$ and $FSI(N) = (S \cup S^{\tau\prime\prime}, O, U'', F'', M_0)$. Let $g \in MP(SI(N))$ with the associated occurrence net $N_g = (S_g \cup S^{\tau}_g, O_g, U_g, F_g, M_{0g})$ where $s \in S^{\tau}_g \Leftrightarrow g(s) \in S^{\tau}$.

Then the net $N_h = (S_g \cup S^{\tau}_h, O_g, U_h, F_h, M_{0g})$ is an occurrence net and the function h based upon it is a process of $FSI(N)$ if defined as follows:

$$
\begin{aligned}
O^{nb} &:= \{t \mid t \in O_g, |\{s \mid (s,t) \in F_g\}| = 1\}\\
S^{\tau}_h &:= S^{\tau}_g \cup \{s_t \mid (s,t) \in F_g, t \in O^{nb}\}\\
U_h &:= U_g \cup \{t_s \mid (s,t) \in F_g, t \in O^{nb}\}\\
F_h &:= (F_g \setminus \{(s,t) \mid (s,t) \in F_g, t \in O^{nb}\})\\
&\quad \cup \{(s,t_s),(t_s,s_t),(s_t,t) \mid (s,t) \in F_g, t \in O^{nb}\}\\
h(x) &:= \begin{cases} g(t)_{g(s)} & \text{if } x = t_s \in U_h \setminus U_g\\ g(s)_{g(t)} & \text{if } x = s_t \in S^{\tau}_h \setminus S^{\tau}_g\\ g(x) & \text{otherwise}\end{cases}
\end{aligned}
$$

First we show that indeed N_h is an occurrence net: Since $(x,y) \in F_h \Rightarrow (x,y) \in F^{+}_g$ we have that $\forall x,y \in S_g \cup S^{\tau}_h \cup O_h \cup U_h.\ (x,y) \in F^{+}_h \Rightarrow (y,x) \notin F^{+}_h$. Furthermore $\forall s \in S_g \cup S^{\tau}_h.\ \{t \mid (t,s) \in F_h\} = \{t \mid (t,s) \in F_g\}$ and thus $\forall s \in S_g \cup S^{\tau}_h.\ |{}^{\bullet}s| \leq 1$ and also $M_{0g} = \{s \mid {}^{\bullet}s = \varnothing\}$. Since $\forall s \in S_g.\ |s^{\bullet}| = 1$ also $\forall s \in S_g \cup S^{\tau}_h.\ |s^{\bullet}| \leq 1$.

Thus N_h is an occurrence net. We now continue by proving that h is indeed a process of $FSI(N)$.

$$\begin{aligned}
h(S_g \cup S_h^\tau) &= g(S_g) \cup \{g(s)_{g(t)} \mid (s,t) \in F_g, t \in O^{nb}\} \\
&\subseteq S \cup S^{\tau\prime\prime} \cup \{s_t \mid (s,t) \in F', |{}^\bullet t| > 1\} \\
&= S \cup S^{\tau\prime\prime} \\
h(O_g) &= g(O_g) \subseteq O \\
h(U_h) &= g(U_g) \cup \{g(t)_{g(s)} \mid (s,t) \in F_g, t \in O^{nb}\} \\
&\subseteq U'' \cup \{t_s \mid (s,t) \in F', |{}^\bullet t| > 1\} \\
&= U'' \\
h(M_{0g}) &= g(M_{0g}) \subseteq M_0
\end{aligned}$$

Let C be a slice of N_h. Assume there exist $p, q \in C, p \neq q$ with $h(p) = h(q)$. If $p = s_t \in S_h^\tau$, let $p' = s$ otherwise let $p' = p$ and similarly for q'. Because $\forall s \in S_g.\ |s^\bullet| \leq 1$ we have that $p' \neq q'$. Since $h(p) = h(q)$ also $h(p') = h(q')$. But $p', q' \in S_g$ and hence $g(p') = g(q')$. Because $(p', p) \in F_h^*$ and $(q', q) \in F_h^*$ we know that neither $(p', q') \in F_g^+$ nor $(q', p') \in F_g^+$. Thus the set $\{p', q'\}$ can be extended to a slice of N_g over which g would not be injective. This would be a contradiction and therefore h must be injective over slices.

h respects the postset of transitions: For every $t \in O_g \cup U_g$

$$h(t^\bullet) = h(\{s \mid (t,s) \in F_g\}) = g(\{s \mid (t,s) \in F_g\}) = \{s \mid (g(t), s) \in F'\} = h(t)^\bullet \ ,$$

whereas for every $t \in U_h \setminus U_g$, say $t = u_p$, we have

$$h({u_p}^\bullet) = h(\{p_u\}) = \{g(p)_{g(u)}\} = {g(u)_{g(p)}}^\bullet = h(u_p)^\bullet \ .$$

h respects the preset of transitions: For every $t \in U_g \cup O_g \setminus O^{nb}$

$$h({}^\bullet t) = h(\{s \mid (s,t) \in F_g\}) = g(\{s \mid (s,t) \in F_g\}) = \{s \mid (s, g(t)) \in F'\} = {}^\bullet h(t) \ ,$$

whereas for $t \in O^{nb}$

$$h({}^\bullet t) = h(\{s_t \mid (s,t) \in F_g\}) = \{g(s)_{g(t)} \mid (s,t) \in F_g\} = \{s_t \mid (s, g(t)) \in F'\} = {}^\bullet h(t)$$

and for $t \in U_h \setminus U_g$, say $t = u_p$, we have

$$h({}^\bullet t) = h(\{u\}) = \{g(u)\} = {}^\bullet g(u)_{g(p)} = {}^\bullet h(t) \ .$$

Additionally h can be extended to a maximal process with the same visible pomset by executing remaining elements of $U'' \setminus U'$.

Conversely let $h \in MP(FSI(N))$. Let the associated occurrence net be $N_h = (S_h \cup S_h^\tau, O_h, U_h, F_h, M_{0h})$.

Then the net $N_g = (S_h \cup S_g^\tau, O_h, U_g, F_g, M_{0h})$ is an occurrence net and the function g based upon it is a maximal process of $SI(N)$ if defined as follows:

$$
\begin{aligned}
U^{nb} &:= \{t \mid t \in U_h, h(t) \in U'' \setminus U'\} \\
S_g^\tau &:= S_h^\tau \setminus \{p \mid (u,p) \in F_h, u \in U^{nb}\} \\
U_g &:= U_h \setminus U^{nb} \\
F_g &:= \Big(F_h \setminus \\
&\qquad \Big(\{(s,u),(u,p) \mid s \in S_h, u \in U^{nb}, p \in S_h^\tau\}) \cup \\
&\qquad \{(p,t) \mid (u,p) \in F_h, u \in U^{nb}, p \in S_h^\tau, t \in O_h\}\Big)\Big) \\
&\qquad \cup \{(s,t) \mid \{(s,u),(u,p),(p,t)\} \subseteq F_h, s \in S_h, t \in O_h, u \in U^{nb}, p \in S_h^\tau\} \\
g &:= h \restriction S_h \cup S_g^\tau \cup O_h \cup U_g
\end{aligned}
$$

From the definition we get $F_g^+ \subseteq F_h^+$, thus F_g^+ has no cycles. Additionally $F_g \cap ((O_g \cup U_g) \times (S_h \cup S_g^\tau)) = F_h \cap ((O_g \cup U_g)(\times S_h \cup S_g^\tau))$ and therefore $\forall s \in S_h \cup S_g^\tau.\ |{}^\bullet s| \leq 1$ and $M_{0h} = \{s | {}^\bullet s = \varnothing\}$. Assume that $\exists s \in S_h \cup S_g^\tau.\ |\{t \mid (s,t) \in F_g\}| > 1$. Clearly, this can only occur due to the last clause of F_g. However the other two clauses would have removed any post-transitions of s before. Hence $\forall s \in S_h \cup S_g^\tau.\ |s^\bullet| \leq 1$.

Therefore N_g is an occurrence net. We now prove that g is a maximal process of $SI(N)$.

$$
\begin{aligned}
g(S_h \cup S_g^\tau) &= h(S_h \cup S_g^\tau) \subseteq S \cup S^\tau \\
g(O_h) &= h(O_h) \subseteq O \\
g(U_g) &= h(U_h) \setminus \{t_s \mid s \in S, t \in O, |{}^\bullet t| > 1\} \subseteq U' \\
g(M_{0h}) &= h(M_{0h}) \subseteq M_0
\end{aligned}
$$

Let C be a slice of N_g. C is then also a slice of N_h because $F_h^+ \cap (S_h \cup S_g^\tau \cup O_h \cup U_g)^2 = F_g^+$. If $p, q \in C, p \neq q, g(p) = g(q)$ then also $h(p) = h(q)$ which is a contradiction since h is injective over slices. Hence g must be injective over slices as well.

g respects the postset of transitions: For every $t \in O_h \cup U_g$

$$g(t^\bullet) = h(t^\bullet) = h(t)^\bullet = g(t)^\bullet \ .$$

g respects the preset of transitions: If $t \in O_h, |{}^\bullet t| > 1 \vee t \in U_g$ then

$$g({}^\bullet t) = h({}^\bullet t) = {}^\bullet h(t) = {}^\bullet g(t) \ .$$

If $t \in O_h, |{}^\bullet t| \leq 1$ then

$$
\begin{aligned}
g({}^\bullet t) =& h(\{s \mid \{(s,u),(u,p),(p,t)\} \subseteq F_h, u \in U_h, p \in S_h^\tau\}) \\
=& \{s \mid (s,h(u)) \in F'', \{(u,p),(p,t)\} \subseteq F_h, u \in U_h, p \in S_h^\tau\}) \ .
\end{aligned}
$$

However every place $s_t \in S^{\tau\prime\prime}$ has exactly one pre-transition, namely t_s, and thus

$$\begin{aligned} &\{s \mid (s, h(u)) \in F'', \{(u,p),(p,t)\} \subseteq F_h, u \in U_h, p \in S_h^\tau\}) \\ =\ &\{s \mid \{(s,u),(u,h(p))\} \subseteq F'', (p,t) \in F_h, u \in U'', p \in S_h^\tau\}) \\ =\ &\{s \mid \{(s,u),(u,p),(p,h(t))\} \subseteq F'', u \in U'', p \in S^{\tau\prime\prime}\}) \\ =\ &\{s \mid (s,h(t)) \in F'\}) = {}^\bullet g(t)\ . \end{aligned}$$

To prove that g is maximal, assume N_g° would enable some transition t in $SI(N)$. If $t \in U$, then ${}^\bullet t \subseteq S$ and the same t would also be enabled in $FSI(N)$ by N_h°, hence $t \in O$. If $|{}^\bullet t| > 1$ again t would be enabled in $FSI(N)$ by N_h°. Let ${}^\bullet t = \{s\} \subseteq N_g^\circ \cap S$. But then either $s \in N_h^\circ$ or $s_t \in N_h^\circ$, and either t_s or t would be enabled in $FSI(N)$ by N_h°, which is a contradiction. Hence g must be maximal. □

The question remains however why $FSA(B)$ is so small. The motivation for branching time equivalences is the implicit assumption that the system under consideration will later be embedded into an environment which might prohibit execution of some actions.

If this embedment is modelled within the net itself however, the net will often cease to be symmetrically asynchronous with respect to linear time as the communication of the net with the environment creates backward branching transitions. Unsurprisingly this happens exactly if the net or the environment is not failures equivalent to its fully symmetrically asynchronous implementation.

This observation hints that our definition of symmetric asynchrony might be a bit off although it gives nicer results. If the communication with the environment is assumed to be synchronous however, the backward branching nature of the communicating transitions poses no problems, and our construction of $SI(N)$ describe the situation.

After the connection between $FSA(B)$ and $SA(B)$ have been cleared up, we now give a semi-structural characterization of $SA(B)$.

Definition 4.4
A net $N = (S, O, \varnothing, F, M_0)$ *has a partially reachable* N iff $\exists t, u \in O.\, t \neq u \wedge {}^\bullet t \cap {}^\bullet u \neq \varnothing \wedge |{}^\bullet t| > 1 \wedge \exists M \in [M_0\rangle_N.\ {}^\bullet t \subseteq M \vee {}^\bullet u \subseteq M$.

Theorem 4.1
A net N without silent transitions is in $SA(B)$ iff N has no reachable N.

Proof "$\Rightarrow$": Suppose $N = (S, O, \varnothing, F, M_0)$ has a reachable N. We will show that $SI(N) \not\approx_F N$. Since N has a reachable N, $\exists t, u \in O.\ t \neq u \wedge {}^\bullet t \cap {}^\bullet u \neq \varnothing \wedge |{}^\bullet t| > 1 \wedge \exists M \in [M_0\rangle_N.\ {}^\bullet t \in M \vee {}^\bullet u \in M$.

Let $p \in {}^\bullet t \cap {}^\bullet u$ and $q \in {}^\bullet t$ with $q \neq p$. Then $p \in M$. Let $SI(N) = (S \cup S^\tau, O, U', F', M_0)$. By Lemma 4.4 there exists a $\sigma \in O^*$ with $M_0 \overset{\sigma}{\Longrightarrow}_N M$.

There are two cases:

Case 1, ${}^\bullet u \subseteq M$: We will show that $<\sigma, \{u\}> \in \mathscr{F}(SI(N))$ but $<\sigma, \{u\}> \neq \mathscr{F}(N)$. Since N has no silent transitions by Lemma 2.1 whenever $M_0 \overset{\sigma}{\Longrightarrow}_N M'$ then $M' = M$. Since $M \xrightarrow{\{u\}}_N$ we have that $<\sigma, \{u\}> \notin \mathscr{F}(N)$.

Let $M_1 \subseteq S \cup S^\tau$ such that $M \xrightarrow{\{t_p\}}_{SI(N)} M_1$ (such an M_1 exists by construction of $SI(N)$). Note that $p \notin M_1$. $SI(N)$ is divergence free by Lemma 4.2. So there exist $M_2, M_3, \ldots, M_n \subseteq S \cup S^\tau$ such that $M_1 \xrightarrow{\tau} M_2 \xrightarrow{\tau} M_3 \xrightarrow{\tau} \cdots \xrightarrow{\tau} M_n \wedge M_n \not\xrightarrow{\tau}$ for some $n \geq 1$. There is no $v \in U'$ with $p \in v^\circ$ by construction of $SI(N)$. Hence $p \notin M_i$ for $1 \leq i \leq n$.

If $|{}^\bullet u| = 1$ then $p \in {}^\circ u$. Otherwise there exists $p_u \in S^\tau$ with $p_u \in {}^\circ u$. In this case also $p_u \notin M_i$ for $1 \leq i \leq n$ by Lemma 4.1 (i) and (iii) as $p_t \in M_i$ for all $1 \leq i \leq n$.

In both cases $M_n \not\xrightarrow{\{u\}}_{SI(N)}$. Hence $<\sigma, \{u\}> \in \mathscr{F}(SI(N))$.

Case 2, ${}^\bullet u \not\subseteq M$: Then ${}^\bullet t \subseteq M$. Thus $\exists q \in {}^\bullet u \setminus {}^\bullet t$ and $|{}^\bullet u| > 1$. This case proceeds as case 1 with the roles of t and u exchanged.

"$\Leftarrow$": We will show that if $SI(N) \not\approx_F N$ then $SI(N)$ has a reachable N. Let $SI(N) = (S \cup S^\tau, O, U', F', M_0)$. If $\mathscr{F}(SI(N)) \neq \mathscr{F}(N)$ then $\mathscr{F}(SI(N)) \setminus \mathscr{F}(N) \neq \varnothing$ by Proposition 4.1. Let $<\sigma, X> \in \mathscr{F}(SI(N)) \setminus \mathscr{F}(N)$. Then $M_0 \overset{\sigma}{\Longrightarrow}_N$ by Lemma 4.5 and Lemma 4.4. Let $u \in X$ such that $M_0 \overset{\sigma u}{\Longrightarrow}$ (which exists, otherwise $<\sigma, X> \in \mathscr{F}(N)$). Let $M_1 \subseteq S \cup S^\tau$ such that $M_0 \overset{\sigma}{\Longrightarrow}_{SI(N)} M_1 \not\xrightarrow{\{u\}} \wedge M_1 \not\xrightarrow{\tau}$ (which exists by Lemma 4.2).

If $|{}^\bullet u| = 1$, let $\{p\} = {}^\bullet u$ and we have $p \notin M_1$ (otherwise $M_1 \xrightarrow{\{u\}}_{SI(N)}$). On the other hand, $M_0 \overset{\sigma u}{\Longrightarrow}_N$ and thus, according to Lemma 4.5 and Lemma 4.4, $p \in \tau^{\leftarrow}(M_1)$. Then, by construction of $\tau^{\leftarrow}$, there must exist some $t_p \in U'$ with $p \in {}^\bullet t$ (which removed the token from p). By the construction of $SI(N)$ then $t \in O$ and, since $|{}^\bullet t| > 1$, also $t \neq u$.

Otherwise $|{}^\bullet u| > 1$. Let $p \in {}^\bullet u$ such that $p \notin M_1 \wedge p_u \notin M_1$ (such p exists, otherwise $M_1 \xrightarrow{\tau}_{SI(N)}$ or $M_1 \xrightarrow{\{u\}}_{SI(N)}$). As above $M_0 \overset{\sigma u}{\Longrightarrow}_N$ and $p \in \tau^{\leftarrow}(M_1)$. Then by construction of $\tau^{\leftarrow}$, either $p \in M_1$, which is not the case, or there exists some $p_t \in M_1$ with $t \in O \wedge p \in {}^\bullet t$. But $p_u \notin M_1$ and hence $t \neq u$.

It follows in both cases that $t, u \in O \wedge t \neq u \wedge {}^\bullet t \cap {}^\bullet u \supseteq \{p\} \wedge |{}^\bullet t| > 1 \wedge M \in [M_0\rangle_N \wedge {}^\bullet u \subseteq M$.
$\square$

It turns out that our net classes $SA(B)$ and $SA(L)$ are strongly related to the following established ones [3].

Definition 4.5 Let $N = (S, O, \varnothing, F, M_0)$ be a net.

(i) N is *free choice*, $N \in FC$, iff $\forall p, q \in S.\ p^\bullet \cap q^\bullet \neq \varnothing \Rightarrow |p^\bullet| = |q^\bullet| = 1$.

(ii) N is *extended free choice*, $N \in EFC$, iff $\forall p, q \in S.\ p^\bullet \cap q^\bullet \neq \varnothing \Rightarrow p^\bullet = q^\bullet$.

(iii) N is *behaviourally free choice*, $N \in BFC$, iff $\forall u, v \in O.\ {}^\bullet u \cap {}^\bullet v \neq \varnothing \Rightarrow (\forall M_1 \in [M_0\rangle.\ {}^\bullet u \subseteq M_1 \Leftrightarrow {}^\bullet v \subseteq M_1)$.

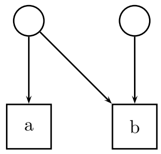

Figure 4.2: $N \in SA(B)$, $N \notin EFC$, $N \notin FC$

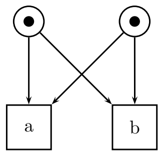

Figure 4.3: $N \in EFC$, $N \notin SA(L)$, $N \notin SA(B)$, $N \notin FC$, $N \notin AA(H,B)$, $N \in AA(M,B)$, $N \in AA(H,L)$, $N \in AA(V,B)$, $N \in ESPL$, $N \notin SPL$, $N \in TSPL$

The class of free choice nets is strictly smaller than that of symmetrically asynchronous nets respecting branching time equivalence.

Proposition 4.3

$$FC \subsetneq SA(B)$$

Proof "$\subseteq$": We prove that if N has a reachable N it is not in FC. Let $t, u \in O$ such that ${}^\bullet t \cap {}^\bullet u \neq \varnothing \wedge |{}^\bullet t| > 1$. Let $p \in {}^\bullet t \cap {}^\bullet u$ and let $q \in {}^\bullet t$ with $p \neq q$. Then $p, q \in S \wedge t \in p^\bullet \cap q^\bullet \wedge |p^\bullet| \geq 2$. Hence N is not in FC.

The inequality follows from the example in Figure 4.2, which is not in FC and trivially in $SA(B)$ as no steps are possible. $\square$

The class of free choice nets is strictly smaller than the class of extended free choice nets.

Proposition 4.4

$$FC \subsetneq EFC$$

Proof Follows from the definitions since $|p^\bullet| = |q^\bullet| = 1 \wedge p^\bullet \cap q^\bullet \neq \varnothing \Rightarrow p^\bullet = q^\bullet$ and the counterexample in Figure 4.3. [3] $\square$

The class of symmetrically asynchronous nets respecting branching time is strictly smaller than the class of symmetrically asynchronous nets respecting linear time.

Proposition 4.5

$$SA(B) \subsetneq SA(L)$$

Proof "$\subseteq$":

We show that $N \notin SA(L) \Rightarrow N \notin SA(B)$.

Let $N = (S, O, \varnothing, F, M_0)$ be a net and $N \notin SA(L)$. From Lemma 4.7 we already know that $MVP(N) \subseteq MVP(SI(N))$. Hence let $f : (S_f \cup O_f \cup U_f) \to (S \cup O \cup U')$ be a maximal process of $SI(N) = (S \cup S^\tau, O, U', F', M_0)$ with $VP(f) \in MVP(SI(N)) \setminus MVP(N)$ based upon an occurrence net $N_f = (S_f, O_f, U_f, F_f, M_{0f})$.

Using f, we will construct a failure of pair of $SI(N)$ which is not a failure pair of N.

Consider the function $g := f \cap ((S_f \cup O_f) \times (S \cup O))$ and the occurrence net defined by $N_g := (g^{\leftarrow}(S), g^{\leftarrow}(O), \varnothing, F_g, g^{\leftarrow}(M_0))$, where

$$(x, y) \in F_g \Leftrightarrow \Big((x, y) \in F_f \vee \exists t \in U_f, s \in S_f, f(s) \in S^\tau.\ \{(x, t), (t, s), (s, y)\} \subseteq F_f\Big)\ .$$

We now show that g is a process of N and $VP(g) = VP(f)$.

From the definition follows directly that

$$\begin{aligned} g(g^{\leftarrow}(S)) &\subseteq S \\ g(g^{\leftarrow}(O)) &\subseteq O \\ g(\varnothing) &= \varnothing \\ g(g^{\leftarrow}(M_0)) &\subseteq M_0\ . \end{aligned}$$

Let $p, q \in S_f$ such that $(p, q) \in F_f^*$. Then by construction of $SI(N)$ there exists a sequence $r_0, r_2, \ldots, r_n$ with $\forall 0 \leq i \leq n.\ f(r_i) \in S$ of places such that

$$\begin{aligned} \forall 1 \leq i \leq n.\ &(\exists t \in O_f.\ \{(r_{i-1}, t), (t, r_i)\} \subseteq F_f) \vee \\ &(\exists t \in O_f, u \in U_f, s \in S_f.\ f(s) \in S^\tau \wedge \{(r_{i-1}, u), (u, s), (s, t), (t, r_i)\} \subseteq F_f) \end{aligned} \tag{4.1}$$

and $r_0 = p \wedge r_n = q$. In other words, there are just these two ways in which two "original" places can be connected in N_f.

Let C be a slice of N_g and $p, q \in C$, $p \neq q$ be two places therein, such that $g(p) = g(q)$. Then C is also a slice of N_f, since $f(C) \subseteq S$ and for every pair of places in C Equation 4.1 holds. Additionally since $C \subseteq S_f$, $f(p) = g(p) = g(q) = f(q)$ which is a contradiction. Hence g is injective over slices.

g respects the post-places of a transition $t \in g^{\leftarrow}(O)$:

$$\begin{aligned} g(t^\bullet) &= g(\{s \in g^{\leftarrow}(S) \mid (t,s) \in F_g\}) \\ &= g(\{s \in g^{\leftarrow}(S) \mid (t,s) \in F_f\}) \\ &= f(\{s \in g^{\leftarrow}(S) \mid (t,s) \in F_f\}) \\ &= \{s \in S \mid (f(t),s) \in F\} \\ &= \{s \in S \mid (g(t),s) \in F\} = g(t)^\bullet \end{aligned}$$

g respects the pre-places of a transition $t \in g^{\leftarrow}(O)$:

If $|{}^\bullet t| = 1$ then

$$\begin{aligned} g({}^\bullet t) &= g(\{s \in g^{\leftarrow}(S) \mid (s,t) \in F_g\}) \\ &= f(\{s \in g^{\leftarrow}(S) \mid (s,t) \in F_f\}) \\ &= \{s \in S \mid (s,f(t)) \in F\}) \\ &= \{s \in S \mid (s,g(t)) \in F\}) = {}^\bullet g(t) \ . \end{aligned}$$

If $|{}^\bullet t| > 1$ then

$$\begin{aligned} g({}^\bullet t) &= g(\{s \in g^{\leftarrow}(S) \mid (s,t) \in F_g\}) \\ &= f(\{s \in g^{\leftarrow}(S) \mid \exists u \in U_f, p \in S_f, f(p) \in S^\tau.\ \{(s,u),(u,p),(p,t)\} \subseteq F_f\}) \\ &= \{s \in S \mid \exists u \in U_f, p \in S_f, f(p) \in S^\tau.\ (s,f(u)) \in F', \{(u,p),(p,t)\} \subseteq F_f\} \ . \end{aligned}$$

However from the construction of $SI(N)$, we have that for each $p_t \in S^\tau$ exists exactly one $v \in U$ with $(v, p_t) \in F'$, namely $v = t_p$. Since f respects the pre- and postconditions of transitions we can continue with

$$\begin{aligned} &\{s \in S \mid \exists u \in U_f, p \in S_f, f(p) \in S^\tau.\ (s,f(u)) \in F', \{(u,p),(p,t)\} \subseteq F_f\} \\ =\ &\{s \in S \mid \exists u \in U, p \in S_f, f(p) \in S^\tau.\ \{(s,u),(u,f(p))\} \subseteq F', (p,t) \in F_f\} \\ =\ &\{s \in S \mid \exists u \in U, p \in S^\tau.\ \{(s,u),(u,p),(p,g(t))\} \subseteq F'\} \\ =\ &\{s \in S \mid (s,g(t)) \in F\} = {}^\bullet g(t) \ . \end{aligned}$$

That $VP(g) = VP(f)$ follows from the definition of F_g and Equation 4.1.

Thus we have that g is a process of N and $VP(g) = VP(f)$. But per assumption $VP(f) \notin MVP(N)$ so g must not be maximal.

Finally we use this property to derive the desired failure pair.

Let $t \in O$ such that ${}^\bullet t \subseteq N_g^\circ$. Such a transitions exists, otherwise g would be maximal.

A linearisation of $VP(g)$ respecting the partial order leads to a trace σ. Additionally $VP(g) = VP(f)$, thus a linearisation of f will result in σ, too. But then $\exists M_1.\ M_0 \overset{\sigma}{\Longrightarrow}_{SI(N)} M_1 \wedge M_1 \overset{\tau}{\not\longrightarrow}_{SI(N)} \wedge M_1 \overset{\{t\}}{\not\longrightarrow}_{SI(N)}$ and $M_0 \overset{\sigma}{\Longrightarrow}_N N_g^\circ \overset{\{t\}}{\longrightarrow}_N$. There from we can conclude that $<\sigma, \{t\}> \in F(SI(N)) \setminus F(N)$ and $N \notin SA(B)$.

The inequality follows from the counterexample in Figure 4.4, the symmetrically asynchronous implementation of which has the additional failure $<x, \{a\}>$. $\square$

The class of extended free choice nets and the class of symmetrically asynchronous nets respecting branching time equivalence are incomparable.

Proposition 4.6

$$EFC \nsubseteq SA(B) \wedge SA(B) \nsubseteq EFC$$

Proof The proposition follows from the counterexamples in Figure 4.2 and Figure 4.3. The latter ones symmetrically asynchronous implementation has the empty pomset as an additional maximal visible pomset and is hence neither in $SA(L)$ nor in $SA(B)$. $\square$

The class of extended free choice nets and the class of symmetrically asynchronous nets respecting linear time equivalence are incomparable.

Proposition 4.7

$$EFC \nsubseteq SA(L) \wedge SA(L) \nsubseteq EFC$$

Proof Again from the counterexamples in Figure 4.2 and Figure 4.3. $\square$

The class of extended free choice nets is strictly smaller than the class of behaviourally free choice nets.

Proposition 4.8

$$EFC \subsetneq BFC$$

Proof We prove $N \notin BFC \Rightarrow N \notin EFC$. Let $N = (S, O, \varnothing, F, M_0)$ be a net. Let $u, v \in O$ with ${}^\bullet u \cap {}^\bullet v \neq \varnothing$. Let $X := {}^\bullet u \cap {}^\bullet v$. Let $M_1 \in [M_0\rangle$ with $\exists M_2.\ M_1[\{u\}\rangle M_2$ and $\nexists M_3.\ M_1[\{v\}\rangle M_3$. Then there is some $p \in {}^\bullet v, p \notin M_1, p \notin {}^\bullet u$. However X is not empty and therefore $\exists q \in X.\ u \in q^\bullet \wedge v \in q^\bullet$. But then $q^\bullet \cap p^\bullet \supseteq \{u\} \neq \varnothing$ and $p^\bullet \neq q^\bullet$ and therefore $N \notin EFC$. The inequality follows from Figure 4.5. [3] $\square$

The class of behaviourally free choice nets and the class of symmetrically asynchronous nets respecting linear time equivalence are incomparable.

Proposition 4.9

$$BFC \nsubseteq SA(L) \wedge SA(L) \nsubseteq BFC$$

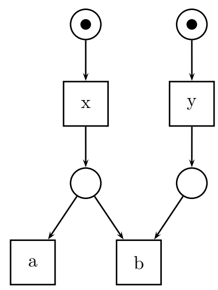

Figure 4.4: $N \in SA(L)$, $N \notin BFC$, $N \notin SA(B)$

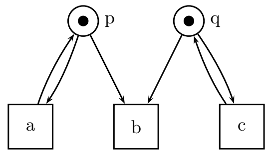

Figure 4.5: $N \in BFC$, $N \notin SA(L)$, $N \notin EFC$

Proof The proposition follows from the counterexamples in Figure 4.4 and Figure 4.5. The latter ones symmetrically asynchronous implementation has an additional maximal process in which b_p fired once and c fires infinitely often. $\square$

The class of symmetrically asynchronous nets respecting branching time equivalence is strictly smaller than the class of behaviourally free choice nets.

Proposition 4.10

$$SA(B) \subsetneq BFC$$

Proof "$\subseteq$": We show that $N \notin BFC \Rightarrow N \notin SA(B)$.

Let $N = (S, O, \varnothing, F, M_0)$ be a net with $N \notin BFC$ and let $SI(N) = (S, O, U', F', M_0)$.

Let $M_1 \in [M_0\rangle, u \in O, v \in O, s \in S$ such that $s \in {}^\bullet v \cap {}^\bullet u \wedge {}^\bullet u \subseteq M_1 \wedge {}^\bullet v \nsubseteq M_1$ (these exist since $N \notin BFC$).

Then there exists a trace σ such that $M_0 \overset{\sigma}{\Longrightarrow}_N M_1$. Together with ${}^\bullet u \subseteq M_1$ it follows that $<\sigma, \{u\}> \notin F(N)$.

Using Lemma 4.4, $M_0 \overset{\sigma}{\Longrightarrow}_{SI(N)} M_1$. Since $s \in {}^\bullet u \subseteq M_1$ but ${}^\bullet v \nsubseteq M_1$ there exists $p \in {}^\bullet v$ with $p \neq s$. Then by construction of $SI(N)$ there exists a transition $v_s \in U'$ (with $s \in {}^\bullet v_s$, $s \notin v_s{}^\bullet$). Thereby $\exists M_2.\ M_1 \xrightarrow{\{v_s\}} M_2$ with $s \notin M_2$. Furthermore $\forall M_3, M_2 \xrightarrow{\tau}{}^* M_3.\ s \notin M_3$ due to the construction of $SI(N)$. Since additionally only finitely many unobservable transitions are possible, $<\sigma, \{u\}> \in F(SI(N))$.

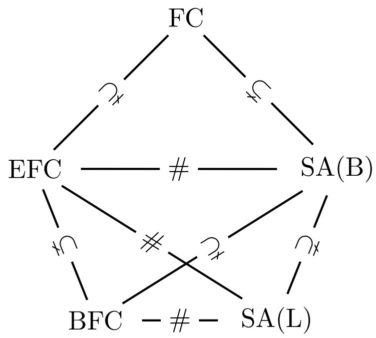

Figure 4.6: Overview of the symmetrically asynchronous net classes

The inequality follows from the counterexample in Figure 4.5, the symmetrically asynchronous implementation of which has the additional failure $<\varepsilon, \{a\}>$. $\square$

The class of fully symmetrically asynchronous nets respecting branching time is strictly smaller than the class of symmetrically asynchronous nets respecting branching time.

Proposition 4.11

$$FSA(B) \subsetneq SA(B)$$

Proof If a net has no partially reachable conflict it also has no partially reachable N.

The inequality follows from the example in figure Figure 3.1. $\square$

We now try to translate our results within Figure 4.6 into intuitive statements about the general nature of asynchrony and synchrony and the implications to the behaviours implementable in an asynchronous system.

Let's start at the top of the diagram, i.e. at FC. Free choice nets are characterized structurally, enforcing that for every place, a token therein can choose freely (i.e. without inquiring about the existence of tokens in any other places) which outgoing arc to take.

This property makes it possible to implement the system asynchronously. In particular, the component which holds the information represented by a token can choose arbitrarily when and into which of multiple asynchronous output channels to forward said information, without further knowledge about the rest of the system. As this decision is solely in the discretion of the sending component and not based upon any knowledge of the rest of the system, no synchronization with other components is necessary.

The difference between $SA(B)$ and FC is that in $SA(B)$ the quantification over the places is dropped, and the condition comes out more straightforward as: Every token can choose freely which outgoing arc to follow. Thus, $SA(B)$ allows for non-free-choice structures as long as these never receive any tokens.

This also explains why BFC includes $SA(B)$. Since $SA(B)$ guarantees that problematic structures never receive any tokens, all transitions contained in such structures are always enabled together (actually they are never enabled).

However $SA(L)$ is not contained in BFC as it additionally allows "temporary" deadlocks which are guaranteed to continue after some further visible behaviour. These kind of later to be resolved deadlocks are forbidden both in branching time semantics and behaviourally free choice nets.

The incomparability between the left and the right side of the diagram stems from the conceptual allowance of slight transformations of the net before evaluating whether it is free choice or not. Specifically in the case of the net in Figure 4.3, a τ transition can first be introduced, which collects both tokens and then produces marks a single post-place from which the two original transitions get the token. Thus the choice between the two transitions is centralized in the newly introduced place and thus free again. We don't allow any insertion of "helping" τ transitions, as it seems unclear to us how much computing power should be allowed in possibly larger networks of such transitions. This becomes especially problematic if these networks somehow track part of the global status of the net inside themselves and thus make quite informed decisions about what outgoing transition to enable.

A similar difference exists between our results and those obtained in [10] by Hopkins. While we enforce a certain distributed implementation of the original net, Hopkins allows any implementation which manages to exhibit the correct visible behaviour. Again, the implementation might be quite elaborate and make informed decisions based upon global knowledge of the net. While such an implementation may be a sensible choice in some cases, it will most likely not be compositional. Since he allows far more transformations than we do and uses interleaving semantics, his net classes include both BFC and $SA(L)$.

5 Asymmetric Asynchrony

As seen in the previous section, the class of symmetrically asynchronous nets is quite small, and precludes the implementation of many real-world behaviours, like waiting for one of multiple input to become readable, a Petri net representation of which will always include non free-choice structures.

Therefore we propose a less strict definition of asynchrony such that actions may depend synchronously on a single predetermined condition. In a hardware implementation the places which earlier could always forward a token into some invisible transitions must now wait until they receive an explicit token removal signal from their post-transitions.

To this end we introduce a static priority over the preplaces of each transition. Every transition first removes the token from the most prioritised preplace and then continues along decreasing priority. To formalize this behaviour in a Petri net we insert an invisible transition for each incoming arc of every transition. These invisible transitions are forced to execute in sequence by newly introduced buffer places between them. Finally at one position in this chain, the original visible transition is inserted.

An example of this transformation is given in Figure 5.1.

Definition 5.1 Let $N = (S, O, \varnothing, F, M_0)$ be a net.
Let $g \subset (S \times O) \times (S \times O)$ be a priority on $F \cap (S \times O)$ such that for each $t \in O$ $g \cap ({}^\bullet t \times \{t\})$ is a total order $\leq_g^t$ over ${}^\bullet t \times \{t\}$.

We write $\min_g^t$ and $\min_g^t$ for the place contained in its minimal and maximal element respectively and $(s+1)_g^t$ for the next place greater than s out of ${}^\bullet t$ according to g.

We define a set of invisible transitions as $X := \{t_s \mid t \in O, s \in {}^\bullet t\}$.

Let $h : X \to X \cup O$ be an injective function for which $\forall t_s \in X.\ h(t_s) \neq t_s \Rightarrow h(t_s) = t$ and $O \subseteq h^{-1}(X)$.

The *asymmetrically asynchronous implementation with respect to g and h* of N is defined as $AI_{g,h} := (S \cup S^\tau, O, U', F', M_0)$ with

$$
\begin{aligned}
S^\tau &:= \{s_t \mid t \in O, s \in {}^\bullet t, s \neq \min_g^t\}\ , \\
U' &:= h(X) \setminus O \text{ and} \\
F' &:= \{(s, h(t_s)) \ \mid t \in O, s \in {}^\bullet t\} \cup \\
&\quad\ \{(p_t, h(t_s)) \mid t \in O, p = (s+1)_g^t, s \in {}^\bullet t, s \neq \min_g^t\} \cup \\
&\quad\ \{(h(t_s), s_t) \mid t \in O, s \in {}^\bullet t, s \neq \min_g^t\} \cup \\
&\quad\ \{(h(t_s), p) \ \mid t \in O, s = \min_g^t, p \in t^\bullet\}\ .
\end{aligned}
$$

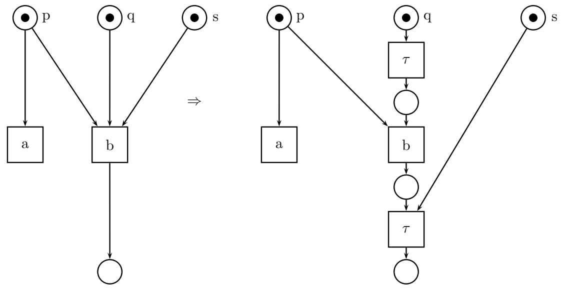

Figure 5.1: Transformation to asymmetric asynchrony, g such that $s <_g^b p <_g^b q$ and h such that $h(a_p) = a, h(b_p) = b, h(b_q) = b_q, h(b_s) = b_s$

Naturally we want the implementation to behave similar to the original net. Contrary to the earlier results and due to the choice of g and h however, it is now possible to create implementations which have additional traces, as it is done in Figure 5.6 by the implementation sketched.

Those problems can be circumvented if h is restricted such that $h(t_s) = t \Rightarrow s = \min_g^t$. Due to time constraints we will consider that restriction to be in place for the most remaining parts and write $AI_g(N)$ instead of $AI_{g,h}(N)$ where it is the case. Additionally we leave the linear time case as conjectures.

We now proceed parallel to the earlier sections, by removing all tokens on S^τ in a marking of the implementation. This time however multiple silent transitions need to be undone in sequence.

Definition 5.2 Let $N = (S, O, \varnothing, F, M_0)$ be a net and $AI_g(N) = (S \cup S^\tau, O, U', F', M_0)$. Let $\tau^\Leftarrow : \mathcal{P}(S \cup S^\tau) \to \mathcal{P}(S)$ be the function defined by

$$\tau^\Leftarrow(X) := (X \cap S) \cup \left\{ s \,\middle|\, \exists t \in O.\ s \in {}^\bullet t \wedge \exists p_t \in X \cap S^\tau.\ (p, t) \leq_g^t (s, t) \right\} .$$

Given a marking of the implementation, $\tau^\Leftarrow$ will produce a marking which must have been reachable before the current situation could ever have arisen.

Note that the application of $\tau^\Leftarrow$ is only meaningful for markings where no two elements of S^τ have originated from the same transition. However implementations of contact free nets produce only markings which fulfil this condition, as we will show below.

We first need to give the necessary invariant predicate and distance function.

Definition 5.3 Let $N = (S, O, \varnothing, F, M_0)$ be a net and $AI(N) = (S \cup S^\tau, O, U', F', M_0)$. The predicate $\gamma \subseteq \mathcal{P}(S \cup S^\tau)$ is defined as $\gamma(M) :\Leftrightarrow \tau^\Leftarrow(M) \in [M_0\rangle_N \wedge \forall p, q \in M.\ p \neq q \Rightarrow \tau^\Leftarrow(\{p\}) \cap \tau^\Leftarrow(\{q\}) = \varnothing$. The function $f : \mathcal{P}(S \cup S^\tau) \to \mathbb{N}$ is defined as $f(M) := |M \cap S|$.

We can now prove basic properties similar (but slightly weaker) to those in Lemma 3.1 and Lemma 4.1.

Lemma 5.1 Let $N = (S, O, \varnothing, F, M_0)$ be a net, $AI_g(N) = S \cup S^\tau, O, U', F', M_0)$ and $M \subseteq S \cup S^\tau$.

(i) $\gamma(M_0)$

(ii) $\gamma(M) \Rightarrow (\exists M' \subseteq S \cup S^\tau.\ M \xrightarrow{\tau}_{AI_g(N)} M' \Rightarrow f(M) > 0)$

(iii) $M[G\rangle_{AI_g(N)} M' \wedge \gamma(M) \Rightarrow \forall t \in G.\ (M \setminus {}^\circ t) \cap t^\circ = \varnothing \wedge \tau^\Leftarrow(M) \xrightarrow{G \cap O}_N \tau^\Leftarrow(M') \wedge \gamma(M')$

(iv) $M \xrightarrow{\tau}_{AI_g(N)} M' \Rightarrow f(M) > f(M') \wedge \tau^\Leftarrow(M) = \tau^\Leftarrow(M')$

(v) $M[G\rangle_N M' \Rightarrow M \xrightarrow{\tau}{}^* \xrightarrow{G} \xrightarrow{\tau}{}^*_{AI_g(N)} M'$

Proof
(i): $\tau^\Leftarrow(M_0) = M_0$ which is trivially reachable. Furthermore $\forall s \in S.\ \tau^\Leftarrow(M_0) = \{s\}$ and hence $\forall s, p \in M_0.\ s, p \in S \wedge s \neq p \Rightarrow \tau^\Leftarrow(\{s\}) \cap \tau^\Leftarrow(\{p\}) = \{s\} \cap \{p\} = \varnothing$.

(ii): Assume $\gamma(M)$ and there exists an $M' \subseteq S \cup S^\tau$ such that $M[t_s\rangle_{AI_g(N)} M'$ with some $t \in O, s \in S$. By construction of $AI_g(N)$ then ${}^\circ t_s \cap S \neq \varnothing$. Hence $f(M) > 0$.

(iii): We first prove that $\forall t \in G.\ (M \setminus {}^\circ t) \cap t^\circ = \varnothing$ and $\gamma(M')$.

Consider any $u \in G \cap U'$. Let $t \in O, s \in S$ such that $t_s = u$. Then $s \in {}^\circ u \wedge s \in M$ and $u^\circ = \{s_t\}$. Then $\tau^\Leftarrow(\{s\}) \cap \tau^\Leftarrow(\{s_t\}) \supseteq \{s\}$. Since $\gamma(M)$ and $s \in M$ then $s_t \notin M$. Hence $(M \setminus {}^\circ u) \cap u^\circ = \varnothing$.

Consider any $u \in G \cap O$. Let s be the single element of ${}^\circ u \cap S$. By construction of $AI_g(N)$ and $\tau^\Leftarrow$, ${}^\bullet u = \tau^\Leftarrow({}^\circ u)$. Since by $\gamma(M)$ it follows that $\tau^\Leftarrow(M) \in [M_0\rangle_N$ and N is contact free, we know that $(\tau^\Leftarrow(M) \setminus {}^\bullet u) \cap u^\bullet = \varnothing$. Additionally $u^\circ = u^\bullet$ and hence $(\tau^\Leftarrow(M) \setminus {}^\bullet u) \cap u^\circ = \varnothing$. Note that ${}^\circ u \cap S = \{s\}$. Since $\gamma(M) \wedge {}^\circ u \subseteq M$ it follows that $M \cap {}^\bullet u = \{s\}$. Thereby $(M \setminus {}^\circ u) \cap u^\circ = \varnothing$.

We now want to prove that $\tau^\Leftarrow(M) \xrightarrow{G \cap O}_N \tau^\Leftarrow(M')$.

$$\begin{aligned}
M' &= (M \setminus \{s \mid s \in {}^\circ t, t \in G\}) \cup \{s \mid s \in t^\circ, t \in G\} \\
&= (M \setminus (\{s \mid t_s \in G \cap U'\} \cup \{q_t \mid t_s \in G \cap U', |{}^\circ t_s| > 1, q = (s+1)^t_g\} \cup \\
&\quad \{\min^t_g \mid t \in G \cap O\} \cup \{q_t \mid t \in G \cap O, |{}^\circ t| > 1, q = (\min^t_g + 1)^t_g\})) \cup \\
&\quad \{s_t \mid t_s \in G \cap U'\} \cup \\
&\quad \{p \mid t \in G \cap O, p \in t^\bullet\}
\end{aligned}$$

Therefore

$$\begin{aligned}
\tau^{\Leftarrow}(M') = \tau^{\Leftarrow}((&M \setminus (\{s \mid t_s \in G \cap U'\} \cup \{q_t \mid t_s \in G \cap U', |{}^{\circ}t_s| > 1, q = (s+1)^t_g\} \cup \\
&\{\min^t_g \mid t \in G \cap O\} \cup \{q_t \mid t \in G \cap O, |{}^{\circ}t| > 1, q = (\min^t_g + 1)^t_g\})) \cup \\
&\{s_t \mid t_s \in G \cap U'\} \cup \\
&\{p \mid t \in G \cap O, p \in t^{\bullet}\}) \\
= \tau^{\Leftarrow}((&M \setminus (\{s \mid t_s \in G \cap U'\} \cup \{q_t \mid t_s \in G \cap U', |{}^{\circ}t_s| > 1, q = (s+1)^t_g\} \cup \\
&\{\min^t_g \mid t \in G \cap O\} \cup \{q_t \mid t \in G \cap O, |{}^{\circ}t| > 1, q = (\min^t_g + 1)^t_g\})) \cup \\
&\{s_t \mid t_s \in G \cap U'\}) \cup \\
&\{p \mid t \in G \cap O, p \in t^{\bullet}\} \\
= \tau^{\Leftarrow}((&M \setminus (\{s \mid t_s \in G \cap U'\} \cup \{q_t \mid t_s \in G \cap U', |{}^{\circ}t_s| > 1, q = (s+1)^t_g\} \\
&\{\min^t_g \mid t \in G \cap O\} \cup \{q_t \mid t \in G \cap O, |{}^{\circ}t| > 1, q = (\min^t_g + 1)^t_g\}))) \cup \\
&\{s \mid t_p \in G \cap U', s \in {}^{\bullet}t \wedge (p,t) \leq^t_g (s,t)\} \cup \\
&\{p \mid t \in G \cap O, p \in t^{\bullet}\}\ .
\end{aligned}$$

Since $\forall t \in G \cap O.\ \min^t_g \in M \wedge (|{}^{\circ}t| > 1 \Rightarrow \exists q \in {}^{\bullet}t.\ q = (\min^t_g + 1)^t_g \wedge q_t \in M)$ by $\gamma(M)$ follows that $\nexists p \in M \exists t \in G \cap O.\ \tau^{\Leftarrow}(\{p\}) \cap \tau^{\Leftarrow}({}^{\circ}t) \neq \varnothing \wedge p \neq \min^t_g \wedge (|{}^{\circ}t| = 1 \vee \exists q \in {}^{\bullet}t.\ q = (\min^t_g + 1)^t_g \wedge p \neq t_q)$. Hence

$$\begin{aligned}
\tau^{\Leftarrow}(M') = (\tau^{\Leftarrow}(&M \setminus (\{s \mid t_s \in G \cap U'\} \cup \{q_t \mid t_s \in G \cap U', |{}^{\circ}t_s| > 1, q = (s+1)^t_g\}))\setminus \\
&(\{\min^t_g \mid t \in G \cap O\} \cup \{q \mid t \in G \cap O, |{}^{\circ}t| > 1, q \in {}^{\bullet}t, q \neq \min^t_g\})) \cup \\
&\{s \mid t_p \in G \cap U', s \in {}^{\bullet}t \wedge (p,t) \leq^t_g (s,t)\} \cup \\
&\{p \mid t \in G \cap O, p \in t^{\bullet}\} \\
= (\tau^{\Leftarrow}(&M \setminus (\{s \mid t_s \in G \cap U'\} \cup \{q_t \mid t_s \in G \cap U', |{}^{\circ}t_s| > 1, q = (s+1)^t_g\}))\setminus \\
&\{q \mid t \in G \cap O, q \in {}^{\bullet}t\}) \cup \\
&\{s \mid t_p \in G \cap U', s \in {}^{\bullet}t \wedge (p,t) \leq^t_g (s,t)\} \cup \\
&\{p \mid t \in G \cap O, p \in t^{\bullet}\}\ .
\end{aligned}$$

Since $t_s \in G \cap U' \Rightarrow (s,t) \leq^t_g (s,t)$ and also $t_s \in G \cap U' \wedge |{}^{\circ}t_s| > 1 \wedge q = (s+1)^t_g \Rightarrow (s,t) \leq^t_g (q,t)$ it follows that

$$\begin{aligned}
\tau^{\Leftarrow}(M') = (&\tau^{\Leftarrow}(M)\setminus \\
&\{q \mid t \in G \cap O, q \in {}^{\bullet}t\}) \cup \\
&\{s \mid t_p \in G \cap U', s \in {}^{\bullet}t \wedge (p,t) \leq^t_g (s,t)\} \cup \\
&\{p \mid t \in G \cap O, p \in t^{\bullet}\}\ .
\end{aligned}$$

By construction of $AI_g(N)$ and $\tau^{\Leftarrow}$ follows that $t_p \in G \cap U' \Rightarrow \{s | s \in {}^{\bullet}t \wedge (p,t) \leq^t_g (s,t)\} \subseteq \tau^{\Leftarrow}(M)$. Hence

$$\tau^{\Leftarrow}(M') = (\tau^{\Leftarrow}(M) \setminus \{p \mid t \in G \cap O, p \in {}^{\bullet}t\}) \cup \{p \mid t \in G \cap O, p \in t^{\bullet}\}\ .$$

Since N is contact free there can be no conflict on post-places of any $t \in G \cap O$. By $\gamma(M)$ follows that $\forall t, u \in G \cap O.\ {}^\bullet t \cap {}^\bullet u = \emptyset$. Hence $\tau^{\Leftarrow}(M) \xrightarrow{G\cap O}_N \tau^{\Leftarrow}(M')$.

We still need to prove that $\forall p, q \in M'.\ p \neq q \Rightarrow \tau^{\Leftarrow}(\{p\}) \cap \tau^{\Leftarrow}(\{q\}) = \emptyset$. Assume the contrary, i.e. there are $p, q \in M'$ with $\tau^{\Leftarrow}(\{p\}) \cap \tau^{\Leftarrow}(\{q\}) \neq \emptyset$. Since $\gamma(M)$ at least one of p and q – say p – must not be present in M.

Assume $p \in S^\tau$. Then there exists $s \in S, t \in O$ such that $s_t = p \wedge t_s \in G$. If $|{}^\circ t_s| > 1$ let $r = (s+1)_g^t$. Otherwise let r be a new and unused element (this avoids trivial case differentiations).

From $\gamma(M)$ and ${}^\circ t_s \subseteq M$ follows $p' \in M \wedge \tau^{\Leftarrow}(\{p'\}) \cap \tau^{\Leftarrow}({}^\circ t_s) \neq \emptyset \Rightarrow p' = s \vee p' = r_t$. Additionally $t_s \in G$ and as $s, r_t \notin t_s{}^\circ$ then $s, r_t \notin M'$. Hence any possibly conflicting q must have been created in the same step by some concurrent transition.

Consider a $v \in G \cap U'$ with $v \neq t_s \wedge \tau^{\Leftarrow}(v^\circ) \cap \tau^{\Leftarrow}(t_s{}^\circ) \neq \emptyset$. By construction of $AI_g(N)$ and $\tau^{\Leftarrow}$ then $\tau^{\Leftarrow}({}^\circ v) \cap \tau^{\Leftarrow}({}^\circ t_s) \neq \emptyset$. But then ${}^\circ v \subseteq M$ violates $\gamma(M)$.

Consider a $v \in G \cap O$ with $\tau^{\Leftarrow}(v^\circ) \cap \tau^{\Leftarrow}(t_s{}^\circ) \neq \emptyset$. Let $p' \in \tau^{\Leftarrow}(v^\circ) \cap \tau^{\Leftarrow}(t_s{}^\circ)$. Then also $p' \in v^\bullet$. By construction of $AI_g(N)$ follows that $\tau^{\Leftarrow}(t_s{}^\circ) = \tau^{\Leftarrow}({}^\circ t_s)$. It hence follows that $p' \in \tau^{\Leftarrow}({}^\circ t_s) \subseteq \tau^{\Leftarrow}(M)$.

By $\tau^{\Leftarrow}(M) \xrightarrow{G\cap O} \tau^{\Leftarrow}(M')$ it follows that ${}^\bullet v \subseteq \tau^{\Leftarrow}(M)$. But $\tau^{\Leftarrow}(M)$ is reachable in N and by contact freeness of N follows that $(\tau^{\Leftarrow}(M) \setminus {}^\bullet v) \cap v^\bullet = \emptyset$. Thereby $p' \in {}^\bullet v$.

If $p' = s$ either $s \in {}^\circ v$ and t_s and v could not fire in the same step, or $\exists p'' \in {}^\circ v.\ p'' \neq s \wedge s \in \tau^{\Leftarrow}(\{p''\}) \cap \tau^{\Leftarrow}(\{s\})$ thereby violating $\gamma(M)$

Otherwise $p' \neq s$. But then $\exists p'' \in {}^\circ v.\ p'' \neq r_t \wedge p' \in \tau^{\Leftarrow}(\{p''\}) \cap \tau^{\Leftarrow}(\{r_t\})$ thereby violating $\gamma(M)$.

Assume $p \in S$. Then there exists $t \in G \cap O$ with $p \in t^\circ = t^\bullet$. However $\gamma(M) \Rightarrow M \in [M_0\rangle_N$ and $\tau^{\Leftarrow}(M) \xrightarrow{G\cap O}_N \tau^{\Leftarrow}(M') \Rightarrow {}^\bullet t \subseteq \tau^{\Leftarrow}(M)$. Since N is contact free, then $(t^\bullet \setminus {}^\bullet t) \cap \tau^{\Leftarrow}(M') = \emptyset$. Therefore from $\tau^{\Leftarrow}(\{q\}) \cap \tau^{\Leftarrow}(\{p\}) = \{p\} \wedge p \in t^\bullet \wedge q \in M'$ follows $p \in {}^\bullet t$. Since p was assumed not be in in M and ${}^\circ t \subseteq M$ it follows that $p \notin {}^\circ t$.

If $q \in S$ then $q = p$ thereby contradicting the assumptions. Hence $q \in S^\tau$ and since $q \in M'$ and $t^\circ \subseteq S$ we know that $q \notin t^\circ$ and q was not produced by t.

Now there exists the possibility that q was produced by some other concurrent transition. Assume first that this is not the case and $q \in M$. Then ${}^\circ t \subseteq M \wedge q \in M \wedge q \notin {}^\circ t \wedge p \in \tau^{\Leftarrow}(\{q\}) \cap \tau^{\Leftarrow}({}^\circ t)$ thereby violating $\gamma(M)$.

Assume now that there exists some $v \in G$ with $q \in v^\circ$. Since $q \notin S$ we know that $v \in U'$. By construction of $AI_g(N)$ then there exists some $q' \in {}^\circ v$ with $p \in \tau^{\Leftarrow}(\{q'\})$. Then ${}^\circ t \subseteq M \wedge q' \in M \wedge q' \notin {}^\circ t \wedge p \in \tau^{\Leftarrow}(\{q'\}) \cap \tau^{\Leftarrow}({}^\circ t)$ thereby violating $\gamma(M)$.

(iv): Let $t_s \in U'$ such that $M[\{t_s\}\rangle_{AI_g(N)} M'$. Then ${}^\circ t_s \cap S = \{s\}$. As $t_s{}^\circ \cap S = \emptyset$ no element of $t_s{}^\circ$ contributes to $f(M')$ and hence $f(M') = f(M) - 1$.

If ${}^\circ t_s \subseteq S$ then $\tau^\Leftarrow(M') = \tau^\Leftarrow((M \setminus {}^\circ t_s) \cup t_s{}^\circ) = \tau^\Leftarrow((M \setminus \{s\}) \cup \{s_t\}) = \tau^\Leftarrow(M)$. Otherwise let $p \in S$ such that $p_t \in {}^\circ t_s$.

Then $\tau^\Leftarrow(M') = \tau^\Leftarrow((M \setminus {}^\circ t_s) \cup t_s{}^\circ) = \tau^\Leftarrow((M \setminus \{s, p_t\}) \cup \{s_t\}) = \tau^\Leftarrow(M)$.

(v): Assume $M[G\rangle_N M'$. Order the elements of G arbitrarily such that $G = \{t_1, t_2, \ldots, t_n\}$. We now construct a sequence $M_1, M_2, \ldots, M_n$ of markings such that ${}^\circ t_1 \subseteq M_1, {}^\circ t_1 \cup {}^\circ t_2 \subseteq M_2, \ldots, {}^\circ t_1 \cup {}^\circ t_2 \cup \cdots \cup {}^\circ t_n \subseteq M_n$. To simplify notation, let $M_0 := M$. To get from M_{i-1} to M_i with $1 \leq i \leq n$ consider the sequence of places $p_1, p_2, \ldots, p_m$ where every $p_j = (p_{j+1}+1)^t_g$ and $p_m = \min^t_g$. Then $M_{i-1}[\{t_{p_1}\}\rangle_{AI_g(N)}[\{t_{p_2}\}\rangle_{AI_g(N)} \cdots [\{t_{p_{m-1}}\}\rangle_{AI_g(N)} M_i$. In this fashion we arrive at M_n. Then $M_n[G\rangle_{AI_g} M''$. By construction of $AI_g(N)$ follows that $M'' = M'$. ☐

We get the same set of corollaries as before.

Lemma 5.2 Let N be a net.

$AI_g(N)$ is divergence free.

Proof By Lemma 5.1 (i), (ii), (iii) and (iv). ☐

Lemma 5.3 Let $N = (S, O, \varnothing, F, M_0)$ be a net.

If N is contact free, so is $AI_g(N)$.

Proof By Lemma 5.1, (i) and (iii). ☐

Lemma 5.4 Let $N = (S, O, \varnothing, F, M_0)$ be a net, $AI_g(N) = (S \cup S^\tau, O, U', F', M_0)$ and $M_1 \in [M_0\rangle_N, M_2 \subseteq S$.

(i) $(M_1 \xrightarrow{G}_N M_2) \Leftrightarrow (M_1 \xrightarrow{\tau}{}^*_{AI_g(N)} \xrightarrow{G}_{AI_g(N)} \xrightarrow{\tau}{}^*_{AI_g(N)} M_2)$

(ii) $(M_1 \overset{\sigma}{\Longrightarrow}_N M_2) \Leftrightarrow (M_1 \overset{\sigma}{\Longrightarrow}_{AI_g(N)} M_2)$

Proof Completely parallel to Lemma 3.4 using Lemma 5.1 instead of Lemma 3.1. ☐

Lemma 5.5 Let $N = (S, O, \varnothing, F, M_0)$ be a net and let $AI_g(N) = (S \cup S^\tau, O, U', F', M_0)$ be an asymmetrically asynchronous implementation of N.

Let $M \subseteq S \cup S^\tau, \sigma \in O^*$ such that $M_0 \overset{\sigma}{\Longrightarrow}_{AI_g(N)} M$ and let $M_S := \tau^\Leftarrow(M)$.

Then $M_0 \overset{\sigma}{\Longrightarrow}_{AI_g(N)} M_S$ and $\nexists M'_S \subseteq S.\ M'_S \neq M_S \wedge M_0 \overset{\sigma}{\Longrightarrow}_{AI_g(N)} M'_S$.

Proof Completely parallel to Lemma 3.5 using Lemma 5.1 instead of Lemma 3.1. ☐

Proposition 5.1 Let $N = (S, O, \varnothing, F, M_0)$ be a net and let g and h be functions such that $AI_g(N) = (S \cup S^\tau, O, U', F', M_0)$ is an asymmetric asynchronous implementation of N and $h(t_s) = t \Rightarrow s = \min_g^t$.

Then $\mathscr{F}(N) \subseteq \mathscr{F}(AI_g(N))$.

Proof Completely analogous to Proposition 3.1 using Lemma 5.1 instead of Lemma 3.1. □

As before, we are interested in the relationship between nets and their possible implementations. The definition of asymmetric asynchrony however allows different implementations for the same net. We define a net to be asymmetrically asynchronous if any of the possible implementations simulates the net sufficiently.

Definition 5.4
The class of *asymmetrically asynchronous nets respecting branching time equivalence* is defined as $AA(M, B) := \{N \mid \exists g, h.\ AI_{g,h}(N) \simeq_F N\}$. Similarly the class of *asymmetrically asynchronous nets respecting linear time equivalence* is defined as $AA(M, L) := \{N \mid \exists g, h.\ AI_{g,h}(N) \simeq_{CPT} N\}$.

These classes can be subdivided further by adding constraints to the possible functions h.

Definition 5.5
The class of *front asymmetrically asynchronous nets respecting branching time equivalence* is defined as $AA(V, B) := \{N \mid \exists g, h.\ h(t_s) = t \Rightarrow s = \min_g^t, AI_{g,h}(N) \simeq_F N\}$. The class of *front asymmetrically asynchronous nets respecting linear time equivalence* is defined as $AA(V, L) := \{N \mid \exists g, h.\ h(t_s) = t \Rightarrow s = \min_g^t, AI_{g,h}(N) \simeq_{CPT} N\}$.

Definition 5.6
The class of *tail asymmetrically asynchronous nets respecting branching time equivalence* is defined as $AA(H, B) := \{N \mid \exists g, h.\ h(t_s) = t \Rightarrow s = \min_g^t, AI_{g,h}(N) \simeq_F N\}$. The class of *tail asymmetrically asynchronous nets respecting linear time equivalence* is defined as $AA(H, L) := \{N \mid \exists g, h.\ h(t_s) = t \Rightarrow s = \min_g^t, AI_{g,h}(N) \simeq_{CPT} N\}$.

We have chosen “V” and “H” from the German “vorne” and “hinten” as “F” for “front” would collide unnecessarily with the “F” of the failure equivalence.

We kindly remind that most of the results in this section only hold for $AA(H, B)$, as we restricted ourselves to it.

It would be nice to obtain a semi-structural characterization of $AA(H, B)$ in the spirit of Theorem 3.1. Unfortunately we did not find exact bounds, but obtained structural upper and lower bounds for that net class.

Definition 5.7

A net $N = (S, O, \varnothing, F, M_0)$ *has a left and right reachable* M iff $\exists t, u, v \in O \exists p \in {}^\bullet t \cap {}^\bullet u \exists q \in {}^\bullet u \cap {}^\bullet v.\ t \neq u \wedge u \neq v \wedge \exists M_1, M_2 \in [M_0\rangle.\ {}^\bullet t \cup {}^\bullet u \subseteq M_1 \wedge {}^\bullet v \cup {}^\bullet u \subseteq M_2$

A net $N = (S, O, \varnothing, F, M_0)$ *has a left and right border reachable* M iff $\exists t, u, v \in O \exists p \in {}^\bullet t \cap {}^\bullet u \exists q \in {}^\bullet u \cap {}^\bullet v.\ t \neq u \wedge u \neq v \wedge \exists M_1, M_2 \in [M_0\rangle.\ {}^\bullet t \subseteq M_1 \wedge {}^\bullet v \subseteq M_2$

Theorem 5.1

If a net $N = (S, O, \varnothing, F, M_0)$ is in $AA(H, B)$ then N has no left and right reachable M.

Proof Assume N has a left and right reachable M. Let $t, u, v \in O$ and $p, q \in S$ such that $p \in {}^\bullet t \cap {}^\bullet u \wedge q \in {}^\bullet u \cap {}^\bullet v \wedge t \neq u \wedge u \neq v \wedge \exists M_1, M_2 \in [M_0\rangle.\ {}^\bullet t \cup {}^\bullet u \subseteq M_1 \wedge {}^\bullet v \cup {}^\bullet u \subseteq M_2$.

The problematic transition will be u. Either $(p, u) >_g^u (q, u)$ or $(q, u) >_g^u (p, u)$. Due to symmetry we can assume the former without loss of generality. We know that there exists some $\sigma \in O^*$ such that $M_0 \overset{\sigma}{\Longrightarrow}_N M_1 \wedge {}^\bullet t \subseteq M_1$. By Lemma 2.1 it follows that $\forall {<}\sigma, X{>} \in \mathscr{F}(N).\ t \notin X$.

By Lemma 5.4 also $M_0 \overset{\sigma}{\Longrightarrow}_{AI_g(N)} M_1$. Let $p_1, p_2, \ldots, p_n \in S$ such that $p_{i-1} = (p_i + 1)_g^u$ for $2 \leq i \leq n$ and $p_n = p$.

Since ${}^\bullet u \subseteq M_1$ then there exists some M_1' with

$$M_1[\{u_{p_1}\}\rangle_{AI_g(N)}[\{u_{p_2}\}\rangle_{AI_g(N)} \cdots [\{u_{p_n}\}\rangle_{AI_g(N)} M_1' \ .$$

Then $p_u \in M_1'$. By Lemma 5.1 (i) and (iii) also $\gamma(M_1')$.

But then by Lemma 5.1 (ii) and (iii) there exists an M_1'' with $M_1' \xrightarrow{\tau}{}^*_{AI_g(N)} M_1'' \wedge M_1'' \overset{\tau}{\not\longrightarrow}_{AI_g(N)} \wedge \gamma(M_1'')$. From construction of $AI_g(N)$ follows $p_u \in M_1' \Rightarrow \exists s \in {}^\bullet u.\ (s, u) \leq_g^u (p, u) \wedge s_u \in M_1''$. By construction of $AI_g(N)$ we know that $p \in \tau^{\Leftarrow}({}^\circ t)$. Together with $\gamma(M_1'')$ follows ${}^\circ t \nsubseteq M_1''$.

But then ${<}\sigma, \{t\}{>} \in \mathscr{F}(AI_g(N))$. By the earlier observation however ${<}\sigma, \{t\}{>} \notin \mathscr{F}(N)$. Hence N is not in $AA(H, B)$. □

Theorem 5.2

If a net $N = (S, O, \varnothing, F, M_0)$ has no left and right border reachable M then N is in $AA(H, B)$.

Proof Assume N has no left and right border reachable M.

Then $\forall u \in O.\ (p, q \in {}^\bullet u \wedge (\exists t \in p^\bullet.\ t \neq u \wedge (\exists M_1 \in [M_0\rangle_N.\ {}^\bullet t \subseteq M_1)) \wedge (\exists v \in q^\bullet.\ v \neq u \wedge (\exists M_2 \in [M_0\rangle_N.\ {}^\bullet v \subseteq M_2))) \Rightarrow p = q$. Hence for every $u \in O$ there can only be one place in ${}^\bullet u$ where conflict could occur.

Now choose $g \subseteq (S \times O) \times (S \times O)$ such that for all $u \in O$, $\min_g^t$ is that single place.

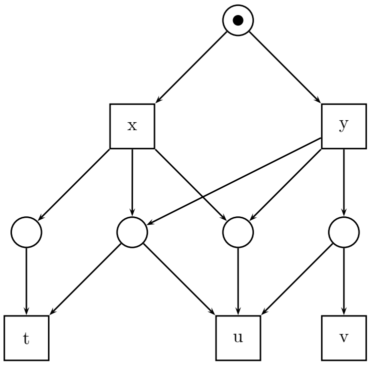

Figure 5.2: $N \notin AA(H,B)$, N has a left and right border reachable M, N has no left and right reachable M

Let $AI_g(N) = (S \cup S^\tau, O, U', F', M_0)$.

We prove that $\mathscr{F}(N) = \mathscr{F}(AI_g(N))$. From Proposition 5.1 we have $\mathscr{F}(N) \subseteq \mathscr{F}(AI_g(N))$. Therefore consider a failure $<\sigma, X> \in \mathscr{F}(AI_g(N))$. We need to show that $<\sigma, X> \in \mathscr{F}(N)$.

There exists some $M_1 \subseteq S \cup S^\tau$ with $M_0 \stackrel{\sigma}{\Longrightarrow}_{AI_g(N)} M_1 \wedge M_1 \stackrel{\tau}{\not\longrightarrow} \wedge \forall t \in X.\ M_1 \stackrel{\{t\}}{\not\longrightarrow}$. Then by Lemma 5.5 $M_0 \stackrel{\sigma}{\Longrightarrow}_{AI_g(N)} \tau^{\Leftarrow}(M_1)$ and by Lemma 5.4 also $M_0 \stackrel{\sigma}{\Longrightarrow}_N \tau^{\Leftarrow}(M_1)$.

Now take any $t \in X$. Assume $\tau^{\Leftarrow}(M_1) \stackrel{\{t\}}{\longrightarrow}_N$. Then $^\circ t \not\subseteq M_1$ but $^\bullet t \subseteq \tau^{\Leftarrow}(M_1)$.

By construction of $\tau^{\Leftarrow}$ then $\forall s \in {}^\bullet t.\ s \in M_1 \vee \exists u \in O, p \in S.\ s \in \tau^{\Leftarrow}(\{u_p\}) \wedge u_p \in M_1$. Since $M_1 \stackrel{\{t\}}{\not\longrightarrow}_{AI_g(N)} \wedge M_1 \stackrel{\tau}{\not\longrightarrow}_{AI_g(N)}$ there exists at least one $s \in {}^\bullet t$ such that $s \neq M_1$ and there exist $u \in O$ and $p \in S$ with $s \in \tau^{\Leftarrow}(\{u_p\})$ and $u \neq t$.

But then $s \in {}^\bullet u \wedge t \in s^\bullet \wedge t \neq u \wedge \tau^{\Leftarrow}(M_1) \in [M_0\rangle_N \wedge {}^\bullet t \subseteq \tau^{\Leftarrow}(M_1)$. Since $s \in \tau^{\Leftarrow}\{u_p\}$ by construction of $AI_g(N)$ follows that $s \neq \min_g^u$. This however contradicts our construction for g given above. Hence $\tau^{\Leftarrow}(M_1) \stackrel{\{t\}}{\not\longrightarrow}_N$.

Applying this argument for all $t \in X$ yields $<\sigma, X> \in \mathscr{F}(N)$ and thereby finally $\mathscr{F}(AI_g(N)) \subseteq \mathscr{F}(N)$. Hence $N \in AA(H,B)$. $\square$

Indeed there are some nets in $AA(H,B)$ which have left and right border reachable Ms, but no left and right reachable M, see Figure 5.2.

As before, the classes defined in this section are related to some known ones.

Definition 5.8 Let $N = (S, O, \varnothing, F, M_0)$ be a net.

(i) N is *simple in terms of transitions*, $N \in TSPL$, iff $\forall u, v \in O.\ ({}^\bullet u)^\bullet \cap ({}^\bullet v)^\bullet \neq \varnothing \Rightarrow {}^\bullet u \subseteq {}^\bullet v \vee {}^\bullet v \subseteq {}^\bullet u$.

(ii) N is *simple*, $N \in SPL$, iff $\forall p, q \in S.\ p^\bullet \cap q^\bullet \neq \varnothing \Rightarrow |p^\bullet| = 1 \vee |q^\bullet| = 1$.

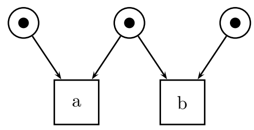

Figure 5.3: $N \in SPL$, $N \notin TSPL$, $N \in AA(H,B)$, $N \in ESPL$

(iii) N is *extended simple*, $N \in ESPL$, iff $\forall p, q \in S.\ p^\bullet \cap q^\bullet \neq \varnothing \Rightarrow p^\bullet \subseteq q^\bullet \vee q^\bullet \subseteq p^\bullet$.

The class of nets which are simple in terms of transitions and simple nets are incomparable.

Proposition 5.2

$$TSPL \nsubseteq SPL \wedge SPL \nsubseteq TSPL$$

Proof The proposition follows from the counterexamples in Figure 4.3 and Figure 5.3. □

The class of nets which are simple in terms of transitions is strictly smaller than the class of extended simple nets.

Proposition 5.3

$$TSPL \subsetneq ESPL$$

Proof Let $N = (S, O, \varnothing, F, M_0)$ be a net. We prove that $N \notin ESPL \Rightarrow N \notin TSPL$.

Let $N \notin ESPL$. Then there exist $p, q \in S$ and $t, u, v \in O$ with $t \in p^\bullet \cap q^\bullet$, $u \in p^\bullet \setminus q^\bullet$ and $v \in q^\bullet \setminus p^\bullet$.

But then $({}^\bullet u)^\bullet \cap ({}^\bullet v)^\bullet \supseteq \{t\}$, yet $\{p\} \in {}^\bullet u \setminus {}^\bullet v$ and $\{q\} \in {}^\bullet v \setminus {}^\bullet u$. Hence $N \notin TSPL$.

The inequality follows from the counterexample in Figure 5.3. □

The class of tail asymmetrically asynchronous nets respecting branching time is incomparable with the class of nets which are simple in terms of transitions.

Proposition 5.4

$$AA(H,B) \nsubseteq TSPL \wedge TSPL \nsubseteq AA(H,B)$$

Proof The proposition follows from the counterexamples in Figure 5.3 and Figure 4.3.

The tail asymmetrically asynchronous implementation of the latter will always have a new failure after the trace ε. If the left token is taken first either a or b will be disabled, but no visible action occurred yet. The same holds for the other side. □

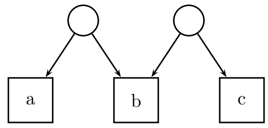

Figure 5.4: $N \in AA(H,B)$, $N \notin ESPL$

The class of simple nets is strictly smaller than the class of extended simple nets.

Proposition 5.5

$$SPL \subsetneq ESPL$$

Proof Let $N = (S, O, \varnothing, F, M_0)$ be a net and $N \in SPL$. If $p^\bullet \cap q^\bullet \neq \varnothing$ then either $|p^\bullet| = 1$, $p^\bullet \cap q^\bullet = p^\bullet$ and $p^\bullet \subseteq q^\bullet$ or vice versa.

The inequality follows from the counterexample in Figure 4.3. □

The class of simple nets is strictly smaller than the class of tail asymmetrically asynchronous nets respecting branching time equivalence.

Proposition 5.6

$$SPL \subsetneq AA(H,B)$$

Proof We prove that every M violates the constraints of SPL.

Assume N has a left and right reachable M. Let $t, u, v \in O$ and let $p, q \in S$ such that $p \in {}^\bullet t \cap {}^\bullet u \wedge q \in {}^\bullet u \cap {}^\bullet v$.

Then $u \in p^\bullet \cap q^\bullet$ and $|p^\bullet| > 1 \wedge |q^\bullet| > 1$. Hence N is not in SPL.

Therefore if N is in SPL it has no M. By Theorem 5.2, N is then in $AA(H,B)$.

The inequality follows from the example in figure Figure 5.4. □

The class of tail asymmetrically asynchronous nets respecting branching time equivalence is incomparable to the class of extended simple nets.

Proposition 5.7

$$AA(H,B) \nsubseteq ESPL \wedge ESPL \nsubseteq AA(H,B)$$

Proof The proposition follows from the counterexamples in Figure 4.3 and Figure 5.4.

The missing tokens in the latter example are intended. As no action is possible there will not be any additional implementation failures. □

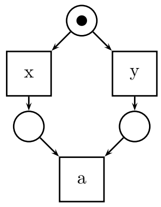

Figure 5.5: $N \notin AA(V, B)$, $N \notin AA(V, L)$, $N \in FC$, $N \in AA(H, B)$

The class of tail asymmetrically asynchronous nets respecting branching time equivalence is strictly smaller than the class of asymmetrically asynchronous nets. While the inclusion is obviously trivial, the inequality is more interesting.

Proposition 5.8

$$AA(H, B) \subsetneq AA(M, B)$$

Proof Follows from the definitions and the counterexample in Figure 4.3.

Every tail asymmetrically asynchronous implementation of the net will have one additional failure, either $<\varepsilon, \{a\}>$ or $<\varepsilon, \{b\}>$. $\square$

Typical nets which are in $AA(M, B)$ but not in $AA(H, B)$ are those with redundant places where it is important to make the choice on the first place taken and do it using a visible transition, lest branching time is violated. However there are less sinister uses of the freedom given in the function h, see Figure 5.6 for an example.

The following result is included merely for sake of completeness, as it is both trivial and rather uninteresting, since the class of front asymmetrically asynchronous nets respecting branching time seems far too small. At least, it's strictly included in the class of asymmetrically asynchronous nets respecting branching time.

Proposition 5.9

$$AA(V, B) \subsetneq AA(M, B)$$

Proof Follows from the definitions and the counterexample in Figure 5.5. In the example, any front asymmetrically asynchronous implementation will have an additional trace, either xa or ya. $\square$

The same relation also holds within linear time semantics.

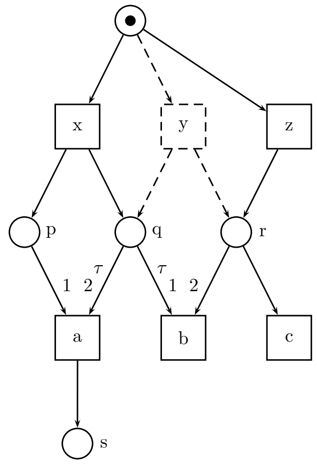

Figure 5.6: $N \in AA(M, L)$, $N \in AA(M, B)$, $N \notin AA(H, L)$

Proposition 5.10

$$AA(V, L) \subsetneq AA(M, L)$$

Proof Follows directly from the definitions and the counterexample in Figure 5.5. $\square$

However some structures are implementable within front asymmetrically asynchronous nets respecting branching time while not in the tail asymmetrically asynchronous variant.

Proposition 5.11

$$AA(H, B) \not\subseteq AA(V, B) \wedge AA(V, B) \not\subseteq AA(H, B)$$

Proof The proposition follows from the counterexamples in Figure 4.3 and Figure 5.5. $\square$

The classes of tail asymmetrically asynchronous nets is strictly smaller than the class of asymmetrically asynchronous nets. This result came quite as a surprise to us and relies heavily upon the fact that we have chosen a behavioural equivalence instead of a notion of simulation which also considers markings.

Proposition 5.12

$$AA(H, L) \subsetneq AA(M, L)$$

Proof $AA(H, L) \subseteq AA(M, L)$ follows directly from the definitions.

The inequality follows from the example in figure Figure 5.6. The dashed parts in the diagram are not necessary for the formal proof, but exist only to highlight the fact that there are such nets where b can be enabled. We prove that no tail asymmetrically asynchronous implementation can be completed pomset trace equivalent to this net (without dashed parts).

The original net has the completed traces zc, and xa. After z, a token resides on r and b must not take that token away, since c must stay enabled until fired. Therefore any implementation of b must first attempt to acquire a token from q. Furthermore after x a token resides on q but b must not fire. Since the token from q must be taken before the one from r, the transition doing so must be invisible. However the trace x must not be maximal but extendible to xa. Since the token on q can be taken away at any time by the invisible transition which is part of the implementation of b, the execution of a must not depend on the existence of a token on q. Hence a must first take the token from p and do so using the visible transition. □

The implementation outlined in the proof of Proposition 5.12 will also work with the dashed parts included, making the example slightly less contrived. Nonetheless, the correctness of the implementation depends crucially on the fact that no further actions get executed after a, as the implementation of a is not guaranteed to run to completion and the place s might not be marked after the trace xa.

This result can be interpreted in two ways. On the one hand, our behavioural approach seems to produce odd results, on the other hand, it identifies special cases which are still implementable by our methods, even though the general structure of them is not.

Those cases which are only implementable by in $AA(M, L)$ are rare however, and we conjecture that the class of tail asymmetrically asynchronous nets is already strictly greater than the class of extended simple nets.

Conjecture 5.1

$$ESPL \subsetneq AA(H, L)$$

Proof (Sketch) Let $N = (S, O, \varnothing, F, M_0)$ be a net and $N \in ESPL$.

We choose g such that $\forall t \in O, p, q \in {}^\bullet t.\ p^\bullet \subseteq q^\bullet \Rightarrow p \leq_g^t q$.

Let $AI_g(N) = (S \cup S^\tau, O, U', F', M_0)$. One needs to show that $MVP(N) = MVP(AI_g(N))$. □

We also conjecture that the class of extended simple nets is strictly smaller than the class of asymmetrically asynchronous nets respecting branching time. First we show a nice property of extended simple nets which can then be used to construct the correct implementation.

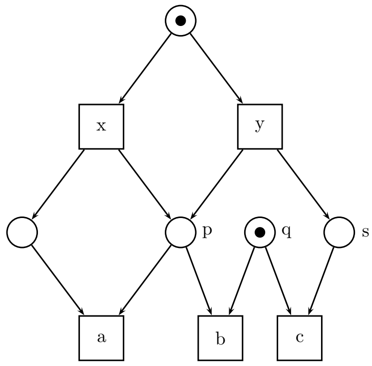

Figure 5.7: $N \in AA(H,L)$, $N \notin AA(M,B)$

Lemma 5.6 Let $N = (S, O, \varnothing, F, M_0)$ be a net with $N \in ESPL$. Let $\# \subseteq O \times O$ be the relation defined as $t\#u :\Leftrightarrow {}^\bullet t \cap {}^\bullet u \neq \varnothing$.

Let $t \in O$. Let $X := \{u \mid t\#^*u\}$. If $|X| > 1$ then $\exists s \in S.\ X \subseteq s^\bullet$.

Proof By induction over the size of a subset Y of X. Begin with $Y := \{t, u\}$ with $t\#u$. By definition of $\#$ there exists an $s \in {}^\bullet t \cap {}^\bullet u \subseteq S$.

Now assume $Y \subseteq X \wedge |Y| > 1$ and there exists an $s \in S$ with $Y \subseteq s^\bullet$. Take a $u \in Y$ and a $v \in X \setminus Y$ with $v\#u$. Then there exists a $p \in {}^\bullet u \cap {}^\bullet v$ by definition of $\#$. But then $s^\bullet \cap p^\bullet \supseteq \{u\}$. Hence either $s^\bullet \subseteq p^\bullet$ or $p^\bullet \subseteq s^\bullet$ by the condition of $ESPL$.

In the first case $Y \cup \{v\} \subseteq p^\bullet$, in the latter case $Y \cup \{v\} \subseteq s^\bullet$. This can be continued until $Y = X$. $\square$

Conjecture 5.2

$ESPL \subsetneq AA(M, B)$

Proof (Sketch) Let $N = (S, O, \varnothing, F, M_0)$ be a net and $N \in ESPL$.

From Lemma 5.6 we get a single dominating preplace for each set of conflicting transitions. We then define g such that $\min_g^t$ is that singe place.

We would need to show that $MVP(N) = MVP(AI_g(N))$. $\square$

We also conjecture that the class of asymmetrically asynchronous nets respecting branching time is strictly smaller than the class of asymmetrically asynchronous nets respecting linear time.

Conjecture 5.3

$$AA(M,B) \subsetneq AA(M,L)$$

Similarly we conjecture that the class of tail asymmetrically asynchronous nets respecting branching time is strictly smaller than the class of tail asymmetrically asynchronous nets respecting linear time.

Conjecture 5.4

$$AA(H,B) \subsetneq AA(H,L)$$

The class of tail asymmetric asynchronous nets respecting linear equivalence is incomparable to the class of asymmetric asynchronous nets respecting branching time equivalence.

Proposition 5.13

$$AA(H,L) \nsubseteq AA(M,B) \wedge AA(M,B) \nsubseteq AA(H,L)$$

Proof By the counterexamples in Figure 5.6 and Figure 5.7. □

The class of symmetrically asynchronous nets respecting branching time equivalence is strictly smaller than the class of asymmetrically asynchronous nets respecting branching time equivalence.

Proposition 5.14

$$SA(B) \subsetneq AA(B)$$

Proof A net which has no partially reachable N also has no left or right border reachable M.

The inequality follows from the example in Figure 4.1. □

Similarly to what we did in Section 4, we now try to translate Figure 5.8 into an intuitive description.

The classes $AA(V,B)$ and $AA(V,L)$ on the right side are as weakly connected as they are since the associated implementations cannot test whether all pre-places of a transition are actually marked, thereby producing additional traces which were not possible in the original net. The resulting net classes are therefore quite small and we didn't think it very important to map their relation to the other classes.

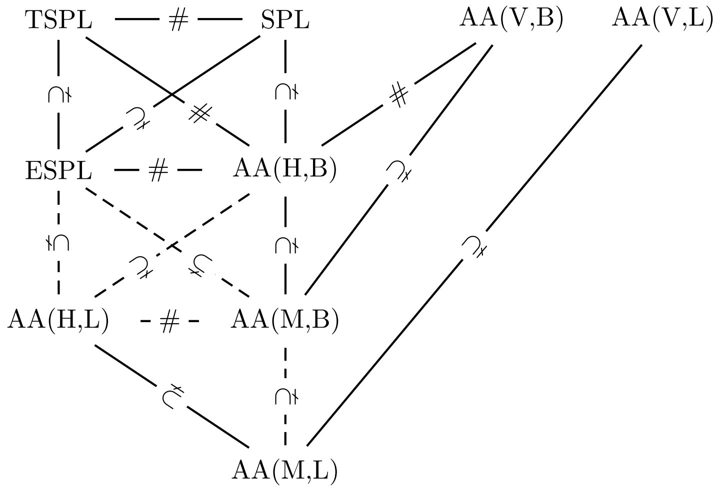

Figure 5.8: Overview of the asymmetrically asynchronous net classes

The inequality between $AA(H, B)$ and $AA(M, B)$ stems from the ability of $AA(M, B)$ to delay removal of tokens until the visible transition has fired. This usually only works when said tokens are guaranteed to stay where they are until the transition fired, a situation commonly encountered when multiple preplaces are common to two transitions. Such nets lie not in $AA(H, B)$ since as soon as the first token on a shared preplace is removed using a silent transition branching time equivalences are violated.

No such problem occurs in linear time however, but unfortunately the power of choosing freely where to insert the visible transition can be used to implement corner cases as the one in Figure 5.6. We don't think there is any meaningful difference between $AA(M, L)$ and $AA(H, L)$ however.

The differences between $AA(M, L)$ and $AA(M, B)$ and between $AA(H, L)$ and $AA(H, B)$ are caused by the possibility of linear time respecting implementations to deadlock temporarily, i.e. a token seems stuck somewhere in the implementation of an transition, but another part of the net continues and finally resolves the deadlock. If the token which seems stuck could have been used by another transition in the original net, such a temporary deadlock violates branching time equivalences, but not linear time equivalences.

Similar to the difference between FC and EFC there exists a difference between $ESPL$ and SPL which originates from the fact that $ESPL$ allows small transformations to a net before testing whether it lies in SPL. This time however our semantically asynchronous classes (aside from $AA(H, B)$) are large enough to contain the untransformed net structure directly, hence the inclusion of $ESPL$ in $AA(H, L)$ and $AA(M, B)$.

6 Conclusions and Related Work

In this paper we have shown how different grades of asynchrony can be modelled in Petri nets. We defined three substantially different families of semantically characterized net classes. In the first family of classes (FSA) it is assumed that removal of tokens happens spontaneously but takes some time to complete. In the second family of classes (SA), these assumptions are held up in principle but transitions which have only one preplace can remove tokens atomically from that single preplace. Finally in the third family of classes (AA) the transitions can control the removal of tokens in so far as tokens are only removed in a static sequence. We have proven a chain of true inclusions between those three families.

Furthermore we have shown which of the known Petri net classes can be implemented using which grade of asynchrony. Specifically we found that free choice nets correspond to the second family of net classes and asymmetric choice nets correspond to the third family.

Similar considerations have already be done in the context of process algebras, mainly π-calculus, locally synchronous systems and hardware implementations.

In [11] Leslie Lamport outlines the basic problem of missing absolute time in a system of communicating processes. He then derives a total ordering of system-wide events which can then be used to solve synchronization problems. He does not detail the implementation of the processes involved in his systems and local synchrony seems to be implied.

In [12] Leslie Lamport considers arbitration in hardware and outlines various arbitration-free "wait/signal" registers. He notes that nondeterminism is thought to require arbitration but no proof is known. He concludes that only marked graphs can be implemented using these registers. Lamport then introduces "Or-Waiting", i.e. waiting for any of two signals, but has no model available to characterize the resulting processes.

The used communication primitives bear a striking similarity to our symmetrically asynchronous nets. While the Petri nets seem to imply nondeterministic choice in the case of forward branching places, this need not be the case. Since the choice in which direction the token moves is made locally it could as well be done deterministically, for example alternating.

In [16] Potop-Butucaru, Caillaud and Benveniste introduce a notion of "weak endochrony" which characterizes locally synchronous components which can be combined without complications into a globally asynchronous system. They then continue to show that weak endochrony is preserved by composition, which they hope will make synthesis of weakly endochronous systems easier.

In [8] Frank S. de Boer and Catuscia Palamidessi consider various dialects of CSP with differing degrees of asynchrony. In particular, they consider CSP without output guards and CSP without any communication based guards. Furthermore they also consider explicitly asynchronous variants of CSP where output actions cannot block, i.e. asynchronous sending is assumed. Our results are related as they provide further detail between $\mathrm{CSP}_\varnothing$ and CSP_I. Interestingly our model seems to have no distinction parallel to the ACSP/CSP boundary.

The one-to-one communication assumption made in [8] when embedding CSP_I into ACSP_I might be related to the boundary between SA and AA as multiple input-guarded receivers together with one sender can still form an M.

In [15] Catuscia Palamidessi shows that some kinds of synchronous communication are impossible in the asynchronous π-calculus, if certain constraints are placed upon the encodings available. In particular she wants encodings to be homomorphic wrt. parallel composition. She then continues to show that symmetric electoral systems cannot be implemented without mixed choice, i.e. the ability to wait both for input and output possibilities at the same time.

In [9] Dianele Gorla investigates different sublanguages of the asynchronous π-calculus which are obtained by allowing different features of communication, namely arity, pattern-matching and fifo-channels. He then proceeds by detailing which encodings between these languages are possible and which are not. He also enforces encodings to be homomorphic wrt. parallel composition, thereby excluding asymmetric encodings.

In [14] Uwe Nestmann gives encodings between various forms of the asynchronous π-calculus. Due to the inherent asymmetry of input and output and because of the use of atomic transmission of values, the π-calculus setting is non-trivially different from out Petri Net based approach. Since our model has static connectivity, it is especially useful for low-level hardware designs.

In [13] Mark Moir describes a communication scheme for a set of processes on a multiprocessor system which want to perform transactional changes to different blocks of shared memory. By clever intermingling of rather low-level lock and higher level transaction management, the proposed scheme enables truly concurrent execution of processes which concurrently read a shared block while ensuring that no two transactions which modify the same block execute in parallel.

In [17] Wolfang Reisig introduces a class of systems which communicate using buffers and where the relative speeds of different components are guaranteed to be irrelevant. The resulting nets are simple nets. He then proceeds introducing a decision procedure for the problem whether a marking exists which makes the complete system live.

The structural net classes we compare our constructions to where all taken from [4], where Eike Best and M.W. Shields introduce various transformations between free choice nets, simple nets and extended variants thereof. They use "essential equivalence" to compare the behaviour of different nets, which they only give informally. Moreover this equivalence is insensitive to divergence, which is also relied upon in their transformations. They then continue to show some conditions under which liveness can be guaranteed for some of the classes.

In [1], Wil van der Aalst, Ekkart Kindler and Jörg Desel introduce two extensions to extended simple nets, by allowing test arcs to violate the ordering of places. This however assumes a kind of "atomicity" of test arcs, which we did not allow in this paper. In particular we don't implicitly assume that a transition will not change the state of a place it is connected to by test arcs, since in case of deadlock, the temporary removal of a token from such a place might not be temporary indeed.

In [10], Richard P. Hopkins introduces the concept of "distributable" Petri Nets, where each transitions and it's preplaces must reside on a single conceptual machine, while the post-places may reside on another one. He then shows which net structures are distributable if additional τ transitions are allowed to be inserted before the visible transitions. The resulting net structures can be understood to be the coarse limit of what we describe in this paper. Our paper fills in much detail which between his classes and free choice nets. Consequently, his paper gives multiple theorem for non-distributability whereas we give the positive results for smaller classes.

He uses interleaving semantics throughout his paper, and as he himself notes, the distributed implementations of some of the example nets behave differently in true concurrency semantics than the original nets, namely they add concurrency in some cases where two transitions share the same preplace which is also a post-place of both by duplicating said place.

Another relevant difference exists between his definitions and ours, namely his classifications are all structural, in the sense that distributability is not a dependent on the initial marking. While he gives the (obvious) extension of distributability which depends on the initial marking, he unfortunately does not give any theorems about it.

In [5] Luc Bougé considers the problem of implementing symmetric leader election in the sublanguages of CSP obtained by either allowing all guards, only input guards or only unguarded choice. He finds that the possibility of implementing it depends heavily on the structure of the communication graphs, while "truly" symmetric schemes are only possible in CSP with input and output guards. These results should be transferable into our framework by relating the class SA to CSP without guarded choice, and the class AA with CSP with only input guarded choice.

Similarly in [6] Luc Bougé improves upon a distributed snapshot algorithm by Chandy and Lamport, adding the possibility to take repeated snapshots and still using only bounded storage. His algorithm ensures non-interference of different snapshot rounds by means of synchronous communication. Indeed his implementation uses input and output guards in the same choice, leading to structures outside of $AA(H, L)$, and is therefore not easily extendible to asynchronous systems.

However there is still much room for research in the topic of asynchronous systems. We conjecture that, even for ready equivalence, it will not be possible to find an equivalent encoding of general synchronous systems into asynchrony, even if symmetry and homomorphism wrt. parallel composition are not required properties of the encoding (work in progress).

However, these restriction seems not to occur in linear time semantics, and an encoding of general Petri nets into some class of asynchronous nets should be possible, if the equivalence is sufficiently coarse. The necessary class of asynchronous nets seems to be still a bit more synchronous than the three classes introduced in this paper (work in progress).

Another interesting problem is to create the connection from our Petri net based model to real hardware. Most probably, the different grades of asynchrony will result in different performance characteristics of their hardware implementations. It might be interesting to create hardware implementations of the various transition types we introduced and benchmark those, but even more interesting it seems would be to apply the knowledge obtained through our models and try to make new more asynchronous chip designs, thereby improving performance.

Furthermore, standard distributed algorithms could be classified by their implementability within the various asynchronous models, thereby creating some common ground between the various concepts of asynchrony occurring in different papers.

Bibliography

[1] W.M.P. van der Aalst, E. Kindler, and J. Desel. Beyond Asymmetric Choice: A note on some extensions. *Petri Net Newsletter*, 55:3–13, 1998. 54

[2] Albert Benveniste, Benoit Caillaud, and Paul Le Guernic. From synchrony to asynchrony. In *International Conference on Concurrency Theory*, pages 162–177, 1999. 1

[3] E. Best. *Structure theory of Petri nets: The free choice hiatus*, pages 168–206. Springer-Verlag, Berlin, 1987. 27, 28, 31

[4] Eike Best and M. W. Shields. Some equivalence results for free choice nets and simple nets and on the periodicity of live free choice nets. In *CAAP '83: Proceedings of the 8th Colloquium on Trees in Algebra and Programming*, pages 141–154, London, UK, 1983. Springer-Verlag. 53

[5] L. Bougé. On the existence of symmetric algorithms to find leaders in networks of communicating sequential processes. *Acta Inf.*, 25(2):179–201, 1988. 1, 54

[6] Luc Bougé. Repeated snapshots in distributed systems with synchronous communications and their implementation in csp. *Theor. Comput. Sci.*, 49:145–169, 1987. 1, 54

[7] G. N. Buckley and Abraham Silberschatz. An effective implementation for the generalized input-output construct of csp. *ACM Trans. Program. Lang. Syst.*, 5(2):223–235, 1983. 1

[8] F. de Boer and C. Palamidessi. Embedding as a tool for language comparison: On the csp hierarchy, 1991. 53

[9] D. Gorla. On the relative expressive power of asynchronous communication primitives. In L. Aceto and A. Ingolfsdottir, editors, *Proc. of 9th Intern. Conf. on Foundations of Software Science and Computation Structures (FoSSaCS'06)*, volume 3921 of *LNCS*, pages 47–62. Springer, 2006. 1, 53

[10] Richard P. Hopkins. Distributable nets. In *Papers from the 11th International Conference on Applications and Theory of Petri Net*, pages 161–187, London, UK, 1991. Springer-Verlag. 1, 34, 54

[11] L. Lamport. Time, clocks, and the ordering of events in a distributed system, 1978. 52

[12] Leslie Lamport. Arbitration-free synchronization. *Distrib. Comput.*, 16(2-3):219–237, 2003. 1, 52

[13] Mark Moir. Transparent support for wait-free transactions. In *Workshop on Distributed Algorithms*, pages 305–319, 1997. 53

[14] Uwe Nestmann. What is a 'good' encoding of guarded choice? *Journal of Information and Computation*, 156:287–319, 2000. 53

[15] Catuscia Palamidessi. Comparing the expressive power of the synchronous and the asynchronous pi-calculus. In *Symposium on Principles of Programming Languages*, pages 256–265, 1997. 1, 53

[16] Dumitru Potop-Butucaru, Benoit Caillaud, and Albert Benveniste. Concurrency in synchronous systems. 1, 52

[17] Wolfgang Reisig. Deterministic buffer synchronization of sequential processes. *Acta Informatica*, 18:115–134, 1982. 53

[18] Wolfgang Reisig. Partial order semantics versus interleaving semantics for csp-like languages and its impact on fairness. In *Proceedings of the 11th Colloquium on Automata, Languages and Programming*, pages 403–413, London, UK, 1984. Springer-Verlag. 6

[19] R. van Glabbeek. The linear time – branching time spectrum i: The semantics of concrete, sequential processes. 2

[20] Rob J. van Glabbeek. The linear time - branching time spectrum II. In *International Conference on Concurrency Theory*, pages 66–81, 1993. 2

[21] Rob J. van Glabbeek and Ursula Goltz. Refinement of actions and equivalence notions for concurrent systems. *Acta Informatica*, 37(4/5):229–327, 2001. 2